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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A QUANTITATIVE EXAMINATION

OF THE RADAR RESOLUTION PROBLEM

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ABSTRACT

In this report, the ability of a radar to resolve overlapping signals is examined. It is shown that the primary limitation on resolvability is the unpredictable difference between the actual received signal and the waveform for which the target receiver is matched (the distortion in the received signal). The effects of signal distortion on the shape of the "ambiguity function" are examined, based upon a wideband random process model for the distortion. The statistics of the ambiguity function side-lobe region, in the presence of the distortion, are related in a simple manner to the statistics of the distortion process. The optimum two-target resolver for the distorted signal is then derived and its performance is examined for two different conditions of a priori knowledge concerning the signal: (1) All parameters of both target returns known exactly, with only the presence or absence of the second target uncertain, and (2) the phase and amplitude of either or both returns unknown, but all other parameters exactly known. These two cases make evident the comparative seriousness of wide-band distortion and lack of a priori knowledge. Probability of detection vs probability of false-alarm curves are derived for both cases.

The performance of two nonoptimum two-target resolvers which are simpler from a circuitry viewpoint than the optimum resolvers is examined. Included in the non-optimum resolver is the usual "matched" (to a single, isolated return) filter. It is quantitatively demonstrated that for nearly complete overlap in time of the received waveforms from the two targets (even though for "compressed" signals the targets may be separated by more than the width of the main lobe of the ambiguity function) the resolvability of a small target in the presence of a large target tends to become independent of the radar sensitivity. Thus, for sufficiently large cross-section difference, resolution can be achieved only by lowering the ambiguity function side-lobe uncertainties, as one would intuitively expect. The optimum vs nonoptimum resolver performances differ primarily in the rate at which targets become resolvable with decreasing overlap.

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I.   INTRODUCTION*

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Perhaps the most vexing problem for the radar designer is obtaining adequate resolution (the detection of targets in the presence of other reflecting objects). For operation at relatively short ranges, the principal resolution problem is usually resolving targets in a background of more or less continuous ground or sea clutter. At longer ranges, the earth's curvature obscures most of the sources of clutter return, and then the resolution problem is to resolve discrete targets from one another, particularly at frequencies above UHF at which aurora and meteors are not important. This report represents an investigation of the discrete target problem.

Impetus for this investigation arises from consideration of several contemporary radar problems which demand the resolution of individual targets of small radar cross sections at ranges of hundreds of miles. Such problems seem inevitably to be complicated further by the fact that the target of interest is much smaller in radar cross section than the interfering target.†

Even a rudimentary consideration of this kind of resolution problem makes it clear that the spread of anticipated cross sections determines the feasibility of different approaches to the problem. If very great differences (several tens of db's) in cross section must be resolved, the designer has little choice. The only radar "dimension" which can be made entirely free of "side responses" is that of range, since both the Doppler response and the angular response of a radar possess "side lobes," and the side lobes of the radar response to a large target may obscure the main response to a small target. The straightforward method of achieving this "absolute" resolution in range is to use a signal consisting of a single short pulse. However, the ability of a radar to detect the presence of a target and to estimate target parameters in a background of white thermal noise is determined by the ratio of the <u>energy</u> E received from the target during one decision-making interval to the noise power per cycle N_o. Obviously, achieving sufficient energy in very short pulses can result in severe demands on radar components.

Furthermore, it is by no means clear that for all radar applications, <u>range</u> is the most useful dimension in which to achieve good resolution. For example, consider the problem of resolving newly injected satellites from their boosters. The impulsive separating mechanism causes the

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two bodies to differ instantaneously in velocity upon separation, while they gradually separate in range. In principle, then, the bodies are potentially resolvable in velocity (Doppler) before they are resolvable in range.

Thus, it is often impractical and/or undesirable to attempt to achieve high resolution only in range by use of short pulses exclusively. The question which naturally arises is how much "limited" (by side-lobe responses) resolution can be obtained in the dimensions in which such side lobes must exist, such as Doppler frequency or angular dimensions. Once the possibility of utilizing "limited resolution" is raised, it is also appropriate to examine the utility of so-called "pulse-compression" techniques,* which create range side lobes in addition to those in the other dimensions.

Therefore, we shall investigate here fundamental and practical limitations upon the degree of "limited" resolution obtainable from a pulsed radar, with emphasis on those using pulse-compression waveforms.

B. The Direction and Scope of this Study

We shall describe two aspects of the limited resolution problem. First, we will try to determine the fundamental limitation on resolution of returns from targets that are sufficiently "close" (in terms of the observable radar coordinates, such as range, Doppler and angular measurement) so that their return signals "overlap." In particular, we will derive, through a simple but informative model, the probabilities that we will correctly or incorrectly announce the presence of a second target in the side-lobe structure of a first.

Investigation into fundamental resolvability specifies an optimum circuit for resolving the targets. However, even in the happily abstract environment of this study, the hardware realization of the optimum resolver must be admitted to be highly complex. Thus, we will also examine the degree to which certain less complicated processing techniques compare to the optimum resolver in performance. Included among the processors studied is the usual "matched" filter, a filter with impulse response which is the undistorted return from a single, isolated target reversed in time.

C. The "Ambiguity Function" and its Relationship to Resolution

Prior to the early 1950's, investigations of the resolving capabilities of radar appear to have been entirely qualitative, and almost exclusively directed at the generation and reception of shorter pulses. For example, examination of the index to the M.I.T. Radiation Laboratory Series reveals only a single reference to target resolution: a brief statement about the importance of range resolution in airborne radar for high-speed aircraft.[4]

Woodward appears to have first explored the more general problem of resolution in more than one dimension, discussing resolution in both range and Doppler. He is generally credited[†] with first pointing out that a fundamental measure of the ability of a radar to resolve signals from two targets with different parameters (specifically range and Doppler, in Woodward's treatment) is the shape of the cross-correlation function of the returns from two targets with differing parameters.

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*A brief description of some typical waveforms for these techniques is given in Sec. I-A; for more information, the reader is referred to the Bibliography (Refs. 1, 2, 3). Such waveforms are characterized by a time duration-bandwidth product much larger than unity (TW ≫ 1, where W is in cycles per second, T in seconds).

† Sec. 7.1 of Ref. 5.

To formally define this function, we assume that the return from a target is representable by the real part of some complex waveform,[†] $\mu(t, \overline{\gamma}_1)$, where generality can be maintained conveniently by regarding $\overline{\gamma}_1$ as an abstract "vector," the components of which represent the effect of the target parameters of Target Number 1 on the signal. For this notation the cross-correlation function[‡] between two targets is defined by

$$\lambda(\overline{\gamma}_1, \overline{\gamma}_2) = \int_{-\infty}^{\infty} \mu(t, \overline{\gamma}_1)\, \mu^*(t, \overline{\gamma}_2)\, dt \quad , \tag{1}$$

where the asterisk denotes the complex conjugate, and the μ's are generally normalized so that $\lambda(\overline{\gamma}_1, \overline{\gamma}_1) = 1$:

$$\int_{-\infty}^{\infty} |\mu(t, \overline{\gamma}_1)|^2\, dt = 1 \quad . \tag{2}$$

This function is known as "Woodward's Ambiguity Function," or "Woodward's Uncertainty Function."

If we let

$$\mu(t, \overline{\gamma}_1) = \mu(t) \quad , \tag{3}$$

for targets differing only in range and Doppler and for narrow-band waveforms, we can approximate[§] $\mu(t, \overline{\gamma}_2)$ as

$$\mu(t, \overline{\gamma}_2) = \mu(t - \tau)\, e^{j\omega_D t} \quad , \tag{4}$$

where τ is related to the range difference R by $\tau = 2R/c$, and ω_D represents the usual narrow-band approximation to the Doppler shift. Then the range-Doppler form of the ambiguity function is

$$\lambda(\tau, \omega_D) = \int_{-\infty}^{\infty} \mu(t)\, \mu^*(t - \tau)\, e^{-j\omega_D t}\, dt \quad . \tag{5}$$

As Woodward indicates, this function plays an important role in resolution questions, because its amplitude expresses the degree to which signals from different targets differ in structure in the dimensions included in the $\overline{\gamma}$ vectors. When $|\lambda|$ is near unity, the received signals are nearly identical in the dimensions concerned, and one would anticipate that when the two signals are immersed in a noise background, distinction would be difficult. Conversely, a value of $|\lambda|$ near zero indicates distinctive signals, and indicates less difficulty in resolution.[¶]

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† Complex functions, denoted primarily by Greek symbols, will be used throughout this report, since the mathematical notation is considerably simplified. When we speak of a function z(t) we imply a function z(t) = x(t) + jy(t), whose real part x(t) is the actual process of concern, and whose imaginary part y(t) is the Hilbert transform of x(t). This particular specification for y(t) renders the spectrum of z(t) single sided [Z(ω) = 0, ω < 0], and allows one to write z(t) modulated by a CW carrier of angular frequency ω₀ simply as z(t) exp [jω₀t]. Other salient features of this notation will be discussed as required; the unfamiliar reader is referred to Woodward,[5] Sec. 2.9; Kelly and Reed,[6] Doob.[7]

‡ The use of the symbol λ to denote this function is by no means universal, many symbolisms are in use. This notation follows Helstrom.[8,9,10]

§ The Doppler shift is actually a spectrum "stretch," rather than a spectrum shift. The latter, however, is generally conceded a satisfactory approximation for narrow-band signals.

¶ Although, as we shall see for $|\lambda|$, the inevitable uncertainty in $|\lambda|$ due to signal distortion and system errors becomes more of a <u>fundamental</u> limitation than $|\lambda|$ itself.

Further discussion of the role of λ in range and Doppler resolution is given in Woodward[5] and Siebert.[11,12] Kelly[13] has explicitly extended the ambiguity function to the dimensions of range, radial velocity and radial acceleration, and Mayo, et al.,[14] have investigated its properties as a function of range, Doppler and two orthogonal angular measurements, thus including certain antenna effects which become important with extremely wide-band signals.

D. Further Studies of the Resolution Problem

In the previously mentioned studies of radar resolution no attempts were made to quantitatively assess the resolution capability of radar (probabilities of detecting a single target in the presence of others). Helstrom[8,9] was the first to attack the problem quantitatively, the later work[9] was more nearly a model of the usual radar situation. He considered the form and performance of the maximum-likelihood detector for the reception of two signals of known separation in time and frequency, but with individual phases and amplitudes unknown, and the time of arrival of the pair unknown. The two signals can be written (in our notation) with Signal 1 expressed as

$$\psi_1(t) = \beta_1 \mu(t - t_o)$$

and Signal 2 as

$$\psi_2(t) = \beta_2 \mu(t - t_o - \tau)\, e^{j\omega_D(t-t_o-\tau)} \quad ,$$

where β_1 and β_2 are complex amplitudes, and β_1, β_2 and t_o are assumed unknown. Helstrom concludes that as $|\beta_1| \to \infty$, the detection of Signal 2 in the presence of Signal 1 becomes insensitive to $|\beta_1|$, and the cost in required increase in returned energy (from Target 2) for a given P_D and P_F is only 1 or 2 db for $|\lambda|^2 \lesssim 0.1$, even though the amplitude of the interference of Signal 1 at the location of Signal 2 $|\beta_1 \lambda|$ is essentially infinite. This result seems rather contrary to experience. As will be proven later, it is a consequence of assuming the form of the signals $\mu(t)$ and their separation to be exactly known — a situation which does not prevail in practice.

Fowle, Kelly and Sheehan[15] discuss a situation in which there are many targets whose ambiguity functions overlap in a manyfold manner. By assuming that the multitarget background resembles Gaussian noise, an expression for the maximum number of targets resolvable (on the average) was derived, in terms of required signal-to-noise ratio and the cross-section spread present.

In a recent paper, Nilsson[16] qualitatively considered the optimum resolving system for a Bayes mean-square error criterion for a situation in which there is a known number n of point targets which differ only in range. He also extended his results to include an unknown number of targets, adopting a general penalty function a_{ij} for guessing i targets when j are present. He makes no calculations of the optimum processor's performance, nor does he attempt to extend results to resolution in dimensions other than range. Preston[17] has also briefly considered a "near-optimum" resolver, but again in only qualitative terms. Thus, we can see that the published quantitative work on this radar resolution problem is quite meager.*

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*Since the original drafting of this report, W.L. Root has also published a quantitative examination of the resolution problem in "Radar Resolution of Closely Spaced Targets," Trans. IRE PGMIL, MIL-6 197 (April 1962).

II. THE PROBLEM MODEL

In this chapter, the considerations that led to the model of the resolution problem that was studied are reviewed. First, the many types of waveforms in use or contemplated are briefly reviewed and their common features indicated. The many possible sources of distortion that exist for such radar signals are pointed out, and a statistical model of their aggregate effect proposed. This signal distortion is then related to the resulting ambiguity function uncertainty, and some simplifying assumptions about the problem model found feasible.

A. Some Characteristics of Common Radar Waveforms

The details of modulation techniques used or proposed for radar differ considerably, but the more widely used techniques can be categorized approximately.

Since we can establish[5,11] that the principal lobe of the range-Doppler ambiguity function is roughly of an extent in time equal to the reciprocal of the bandwidth[*] $1/W$ and of an extent in Doppler frequency roughly equal to the reciprocal of the signal duration $1/T$, we see that the so-called "duration-bandwidth" product TW is significant. For example, TW approximately specifies the ratio of the transmission duration to the extent in time of the main lobe of the ambiguity function (the "pulse-compression ratio").

We will concentrate our attention on signals of large TW product. It is convenient to classify such signals by whether the bandwidth is achieved by modulation within one long continuous envelope, or by using a number of individual pulses. The former class will be referred to as "coded-pulse" signals; the latter as "pulse-train" signals. Examples of coded-pulse signals are:

(1) Pulses with frequency modulation[3,2] within the pulse, as indicated in Fig. 1(a), with achievable TW products limited only by hardware constraints.

(2) The so-called "phase codes," which make use of discretely switched phase within the pulse envelope, of which perhaps the best known is the "Barker"[18] code of order 13, as indicated in Fig. 1(b), with TW ≈ 13. Greater TW products are possible, of which the so-called "shift-register" codes are an example.

Typical of pulse-train signals are those which are:

(1) Regularly spaced, equal-width pulses, as in Fig. 1(c),[11] or

(2) Trains with nonuniform spacing, width, etc.[19]

We might vary any of the potential train parameters in "pulse train" signals, but hardware considerations usually lead to varying only the interpulse spacing D_i as in Fig. 1(d).

Combinations of the various types may also have desirable properties for some applications as well, of course, but will not be explicitly considered here.

Note that the signals of these classes used in radar are of a binary nature; the amplitude is either zero or a constant nonzero value, due to the difficulty in making efficient radar transmitters which operate as linear amplifiers (high efficiency requires operation in a saturated and therefore constant amplitude mode). This fact is of analytic utility and will be used in the analysis to follow.

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[*]W is in cycles per second throughout this report.

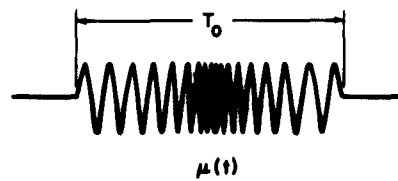Fig. 1(a). Frequency-modulated pulse.

$\mu(t)$

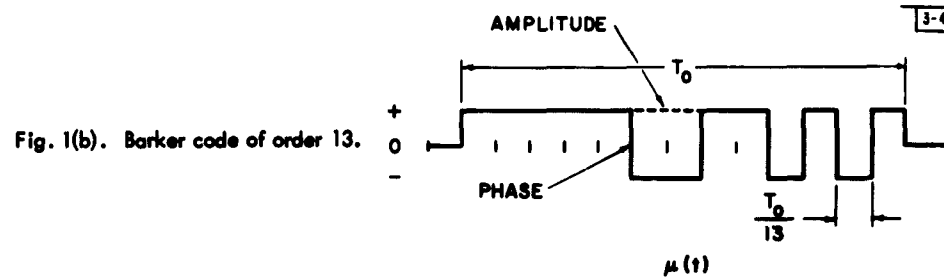Fig. 1(b). Barker code of order 13.

$\mu(t)$

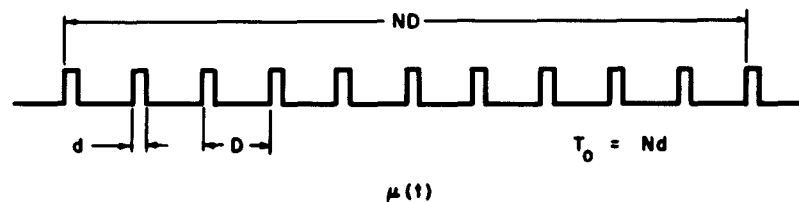$T_0 = Nd$

$\mu(t)$

Fig. 1(c). Regularly spaced pulse train.
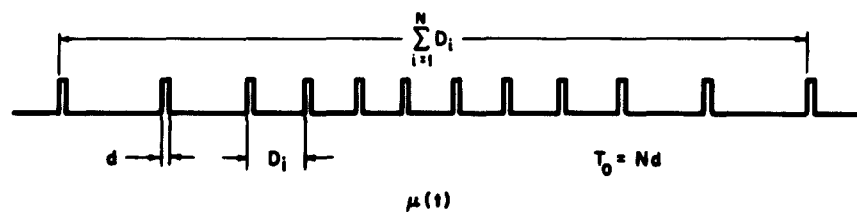

$\sum_{i=1}^{N} D_i$

$T_0 = Nd$

$\mu(t)$

Fig. 1(d). Irregularly spaced pulse train.

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B. Distortion of Radar Signals — Causes and Effects

Helstrom's[9] and Nilsson's[16] results indicate that if one knows the form of the return signal (and, consequently, the shape of the ambiguity function) <u>exactly</u>, the effects of the ambiguity function side lobes of a large target can be <u>completely removed</u> by noting the target "position" and subtracting an appropriately positioned and weighted ambiguity function. However, any uncertainty in the form of the received signal, resulting in an uncertainty in the shape of the ambiguity function,* partly nullifies this cancellation. Furthermore, since the size of the effect of the target is given by the <u>product</u> of its cross-section magnitude and ambiguity function uncertainty (recall that the ambiguity function is normalized), no matter how small the uncertainty in the ambiguity function, a large enough cross section exists to render its effect significant.

Thus, we must be concerned with the effects of even very small ambiguity function uncertainties (values of perhaps −30 db or so being "state of the art"), and consequently, the possibility of even small distortions of the signal returned to the radar as well.
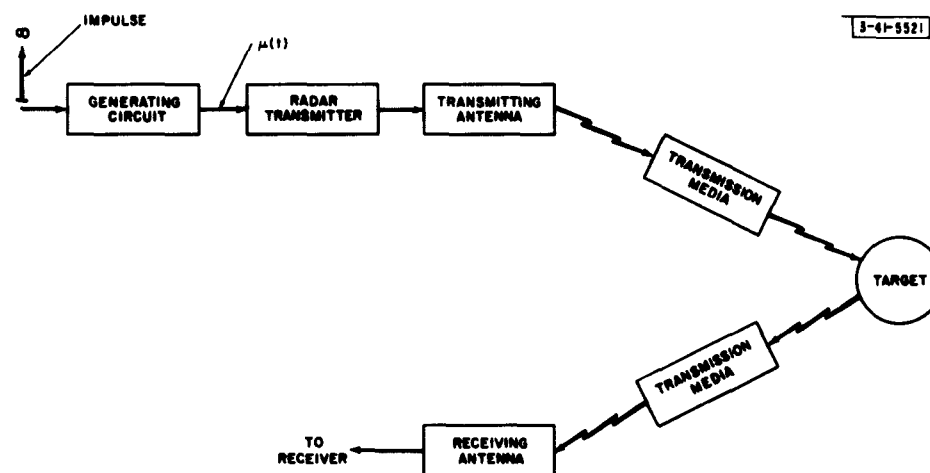
Fig. 2. Conceptual diagram of a typical radar situation.

The possible sources of signal distortion are many. To appreciate this fact, we need only consider the route of a radar signal from the generation of the modulation, to the target, and back to the radar receiver, as indicated in Fig. 2. With no attempt at being exhaustive, we can enumerate some typical distortion sources.

(1) Both the phase and amplitude of the transfer coefficients of the transmitter and receiver may not be linear or flat, respectively. The form of the distortion will, furthermore, have a time-varying component due to such causes as thermal effects and power supply variations.

(2) The frequency characteristics of the transmitting and receiving antennas as "seen" by the target depend not only upon the feed network characteristics, but also upon the angle of the target from the pointing direction of the antenna,[20,14] due to the finite propagation time of energy across the antenna from angles off broadside. Further, the characteristics will depend upon the antenna pointing angle, relative to some fixed reference, due to such causes as rotary joint characteristics,

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* Note that such "uncertainty" may be either truly a matter of ignorance, or merely a matter of convenient neglect of available information.

reflections from nearby objects, and with phased array antennas, the so-called "build-up" phenomena (Part 3, Ch. 2 of Ref. 21), due to the finite time of wavefront propagation across the array.

(3) The effects of the target on the radar waveform are many and varied. First, the Doppler shift introduced by the component of the target motion relative to the radar will actually be a spectrum "stretch," rather than the spectrum displacement usually assumed; the reflected frequency ω_r is related to the instantaneous incident frequency ω_i and the ratio of the velocity component in the radar direction* v_r to the velocity of propagation c by

$$\omega_r = \omega_i \frac{1 - v_r/c}{1 + v_r/c}$$

or, to a first order,

$$\omega_r = \omega_i \left[1 - \frac{2v_r}{c}\right] \quad . \tag{6}$$

The simple constant displacement frequently used for analytic purposes is only accurate for very narrow-band signals.

$$\omega_r = \omega_i + \omega_D \quad , \tag{7}$$

where ω_D is assumed to be a constant related only to the <u>carrier</u> frequency ω_o

$$\omega_D = \frac{2v}{c} \omega_o \quad .$$

Furthermore, v_r/c can be taken as constant only if the transmission time is very short compared to any target acceleration. Finally, if the time extent of the target is of a dimension comparable to the reciprocal of the signal bandwidth and nonsymmetrical as viewed from the radar, the target cross section may vary in both phase and amplitude with frequency over the signal bandwidth.

Not only are there many sources of distortion in the path of a radar signal, but some of these sources are capable of producing distortion of a highly structured, wide-band nature. If the signal is narrow-band, the receiving filters will remove most of the distortion structure from the signal. However, for resolution at long ranges, we are interested in wide-band signals, and must consequently be prepared for similar distortion.

An attempt to explicitly describe the effects of the many possible distortions is difficult. Furthermore, since many of the sources of signal distortion are of a random nature, the use of a statistical model suggests itself. The fact that we expect a time structure to the distortion, leads us to consider the distorted signal from the ith target to be of the form

$$\psi(t, \overline{\tau}_i) = [\beta_i + \delta_i(t)] \mu(t, \overline{\tau}_i) \quad , \tag{8}$$

where β_i is the complex target cross section and $\delta_i(t)$ is a continuous parameter random process (Ref. 22, Sec. 3.7); sample values of $\delta(t)$ taken at some time intervals, $1/W$, are describable in terms of probability by a joint probability density function $p[\delta(1/W), \delta(2/W), \ldots, \delta(n/W), \ldots]$. Furthermore, since we have acknowledged several sources of distortion, the Central Limit Theorem[23] suggests that the real and imaginary parts of $\delta(t)$ will tend toward a complex Gaussian process.[6]

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*v_r taken as positive directed <u>away</u> from the radar.

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We will also assume that the process is stationary in time for a Gaussian process. This requires only that the autocorrelation of samples $\delta(n/W)$ and $\delta(m/W)$, defined as

$$\mathcal{R}(\tfrac{n}{W}, \tfrac{m}{W}) = \overline{\left(\delta^*(\tfrac{n}{W}) - \bar{\delta}^*(\tfrac{n}{W})\right)\left(\delta(\tfrac{m}{W}) - \bar{\delta}(\tfrac{m}{W})\right)} \tag{9}$$

(where the bar denotes the expectation or ensemble average of the process), depend only upon the difference $(n-m)/W$, and not upon the absolute values of n or m.

The choice of the distortion model of the form of Eq. (8) implies that the distortion vanishes with the signal.[†] Furthermore, bearing in mind that $\delta(t)$ is complex the multiplicative form for the distortion proposed indicates that the distortion amplitude multiplies the undistorted signal amplitude, while the phases add.

The signal form of Eq. (8) will therefore be used throughout this report, with $\delta(t)$ taken as a stationary complex Gaussian process.

C. The Alternative Hypotheses

We choose to frame the resolution question by deciding whether or not a second signal is present in the side lobe region of a first. Formally, we are going to choose between two hypotheses regarding a sample of received complex waveform, $\zeta(t)\,e^{j\bar{\omega}_0 t}$. Specifically, we wish to determine whether $\zeta(t)$ is from

$$\zeta(t) = \nu(t) + \psi(t, \gamma_1) \tag{10}$$

or

$$\zeta(t) = \nu(t) + \psi(t, \bar{\gamma}_1) + \psi(t, \bar{\gamma}_2) \quad, \tag{11}$$

where $\nu(t)$ is an additive noise background. The noise will be assumed to be a complex Gaussian process of mean zero and uniform spectral density ("white" noise) of $2N_0$ watts/cycle, for positive frequencies (if one physically measured the noise power density in the real process in a band Δf, the result would be $N_0 \Delta f$).

Equation (8) appears to ignore the fact that $\delta_i(t)$ will usually be related to β_i: the larger the target return, the larger (in terms of probability at least) the distortion. Consequently, we will relate the variance of δ_i to the value β_i (or, if β_i is random, to its variance). Noting that the normalization constraint of Eq. (2) relates the "no-error" cross-section β_i to the energy E_i in the real part of the return from the ith target by the factor

$$\frac{|\beta_i|^2}{2} = E_i \quad, \tag{12}$$

the variance in $\delta_i(t)$ will be taken to be related to E_i by

$$\overline{|\delta_i(t)|^2} = 4E_i \sigma_\delta^2 \quad, \tag{13}$$

so that for unit cross section, the variance of the real part of the distortion is σ_δ^2. We assume, for the present, that E_i is a known constant (in later sections, E_i will be treated as a random variable and we will replace E_i by $\bar{E}$ in the equation). The mean of $\delta_i(t)$ will be taken as zero.

[†] Multipath returns can be considered as separate signals if the path differences are large enough to render the above assumption seriously erroneous.

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While we will investigate the effect of letting β_i be a random variable, it will be assumed throughout the report that $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are known exactly. This assumption is not really as restrictive as it might be at first assumed, at least from a comparative standpoint. It is well known that if the problem is strictly one of detection, the optimum detector for processes involving unknown target parameters can be theoretically realized as the weighted sum of the outputs of a bank of individual filters. Each filter of the bank is matched to a different combination of target parameters. Thus, the heart of any processor for signals with unknown parameters is the processor for a single set of parameters.

However, an apparently important distinction lies in the fact that, for a single target processor, we need only test all possible combinations of parameters, while for even pair-wise resolution, we must examine all possible pairs of combinations. Thus, pair-wise resolutions necessitate combining the outputs of approximately $n^2/2$ filters for large n, while the signal target processor necessitates combining only n. The price we pay in increased signal energy required for this increase in number of filters appears to be small,[†] however, if we can draw an analogy with analyses[24,11] of optimal detection of one of M possible single signals. For example, Siebert's[11] analysis of the increase in E/N_o required for detecting one of M possible orthogonal signals indicates that the increase in E/N_o required with M is

$$\Delta\left(\frac{E}{N_o}\right) \approx \frac{1}{2}\ln M \quad . \tag{14}$$

Bearing in mind that the value of E/N_o which is required to detect a target whose positional parameters are known exactly, is of the order of 25, and that, for example, $\ln 20{,}000 \approx 10$, it is apparent that we pay little penalty for dealing with one of M, or indeed one out of M^2 targets for reasonable M. Consequently, we will restrict further attention to the "known position" case.

In summary, we will concentrate our attention on the pair-wise resolution of targets whose positions are known exactly. Only the distortion of the signals will be assumed unknown throughout the entire report, although the complex target cross section will be assumed unknown in certain sections.

D. The Effects of Signal Distortion on the Ambiguity Function

Before considering the analysis of the central problem, we will examine the effect of the signal distortion on the structure of the ambiguity function.

For distorted signals of the form of Eq. (8), it is useful and seems appropriate to define[‡] the ambiguity function as the cross correlation between a distorted signal and an undistorted reference (thus attaching the physical significance that the ambiguity function so defined is the response of a receiver to a distorted signal with parameters $\overline{\gamma}_1$, when the receiver is matched to an undistorted signal with parameters $\overline{\gamma}_2$):

$$\lambda(\overline{\gamma}_1, \overline{\gamma}_2) = \lambda_{12} = \frac{1}{\beta_1}\int_0^T [\beta_1 + \delta_1(t)]\,\mu_1(t)\,\mu_2^*(t)\,dt \quad , \tag{15}$$

[†] Although the price we pay in hardware complexity certainly may not be small.

[‡] It will be assumed throughout the report that the interval 0 to T is long enough to contain all signal structure of interest. It is long enough to justify the replacement of the limits 0 and T on integrals by $-\infty$ and $+\infty$ with negligible error.

where the factor $1/\beta_1$ normalizes λ in the absence of errors for known β_1. From the properties of $\delta_1(t)$, it follows that the mean of the ambiguity function is its "no-error" form

$$\overline{\lambda}_{12} = \int_0^T \mu_1(t)\,\mu_2^*(t)\,dt \tag{16}$$

and its variance is (recalling that the real part of λ is of concern, and that the variance of the real part σ_λ^2 is one-half the variance of the complex λ)

$$\overline{(\lambda_{12}^* - \overline{\lambda}_{12}^*)\,(\lambda_{12} - \overline{\lambda}_{12})} = 2\sigma_\lambda^2$$

$$= \frac{1}{|\beta_1|^2} \iint_0^T \overline{[\beta_1^* + \delta_1^*(t_1)]\,[\beta_1 + \delta_1(t_2)]}\,\mu_1^*(t_1)\,\mu_2(t_1)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2$$

$$- \iint_0^T \mu_1^*(t_1)\,\mu_2(t_1)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2 \tag{17}$$

which, by virtue of zero mean assumed for $\delta_1(t)$, gives

$$2\sigma_\lambda^2 = \iint_0^T \frac{\overline{[\delta_1^*(t_1)\,\delta_1(t_2)]}}{|\beta_1|^2}\,\mu_1^*(t_1)\,\mu_2(t_1)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2 \quad . \tag{18}$$

In agreement with Eq. (13) for $\overline{|\delta_i(t)|^2}$, we will set

$$\overline{\delta_1^*(t_1)\,\delta_1(t_2)} = 4\sigma_\delta^2 E_1 \rho(t_1 - t_2) \tag{19}$$

for E_1 known and constant, where ρ is the normalized $[\rho(0) = 1]$ autocorrelation function of the errors. Thus, we have, recalling that $E_1 = |\beta_1|^2/2$,

$$\sigma_\lambda^2 = \sigma_\delta^2 \iint_0^T \mu_1^*(t_1)\,\mu_2(t_1)\,\rho(t_1 - t_2)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2 \quad . \tag{20}$$

Let us attempt to attach some physical significance to Eq. (20). Note that we can physically generate the ambiguity function by the use of a linear filter of impulse response $h(t) = \mu^*(-t, \overline{\gamma}_2)\,e^{-j\omega_0 t}$, as indicated in Fig. 3, if the interval 0 to T is sufficiently long [a finite delay may be necessary to realize $h(t)$]. It follows from the properties of linear filters acting on a

$$\left[1 + \frac{\beta_1(t)}{\beta_1}\right]\mu(t,\overline{\gamma}_1)e^{i\omega_0 t} \longrightarrow \boxed{\begin{array}{c}\text{LINEAR FILTER}\\[4pt] h(t) = \mu^*(-t,\overline{\gamma}_2)e^{-i\omega_0 t}\end{array}} \longrightarrow e_0(\tau) = e^{-i\omega_0 \tau}\int_{-\infty}^{\infty}\left[1 + \frac{\beta_1(t)}{\beta_1}\right]\mu(t,\overline{\gamma}_1)\mu^*(t-\tau,\overline{\gamma}_2)\,dt$$
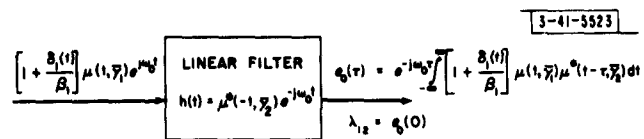
$$\lambda_{12} = e_0(0)$$
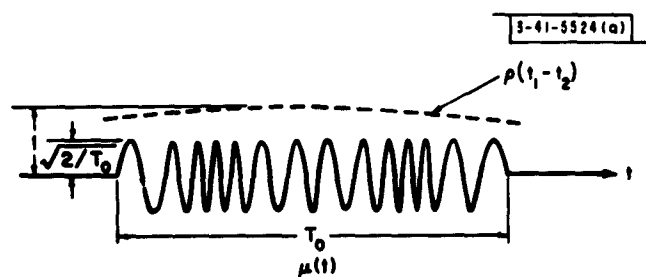
Fig. 3. Linear network for generating λ_{12}.

Fig. 4(a). Comparative shapes of $\mu(t)$ and $\rho(t_1 - t_2)$ for narrow-band distortion of coded pulses.
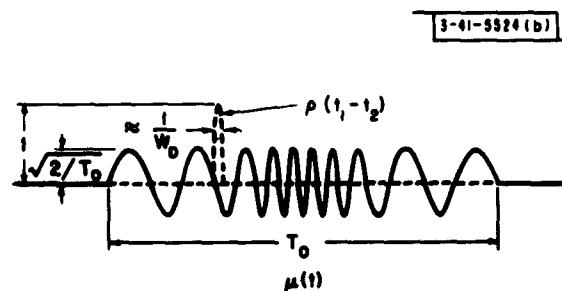
Fig. 4(b). Comparative shapes of $\mu(t)$ and $\rho(t_1 - t_2)$ for wide-band distortion of coded pulses of large-duration-bandwidth product.

complex Gaussian process of this type (the real and imaginary parts of $\delta(t)$ are independent, with equal variances) that the output of the filter shown is Gaussian, with a mean value which is the "no-error" ambiguity function, given by Eq. (16), and a real part variance σ_λ^2, given by Eq. (20).

This result makes possible a useful, practical interpretation of a rather involved integral which will appear frequently in the ensuing analyses. Since λ_{12} is a complex Gaussian process, with mean value $\bar{\lambda}_{12}$, the amplitude $|\lambda_{12}|$, is distributed according to a "Rice"[25] probability density function.†

$$p(|\lambda_{12}|) = \frac{|\lambda_{12}|}{2\sigma_\lambda^2} \exp\left[-\frac{|\lambda_{12}|^2 + |\bar{\lambda}_{12}|^2}{2\sigma_\lambda^2}\right] I_0\left(\frac{|\lambda_{12}| \, |\bar{\lambda}_{12}|}{\sigma_\lambda^2}\right).$$

where $I_0(x)$ denotes the zero-order Bessel function of the first kind of imaginary argument. It then follows from the properties of this function that at points where the "no-error" ambiguity function is zero (or negligible compared to the error), the average amplitude of the ambiguity function in the presence of errors is just $\sqrt{2}\,\sigma_\lambda$. Thus, inspection of the structure of λ from a radar allows a physical estimate of σ_λ.‡

While the foregoing fact gives physical identity to the integral relation of Eq. (20), it does not lend any insight into the relationship between σ_δ and σ_λ. For specified signals and autocorrelation functions, the integral can, at least numerically, be evaluated with precision. Furthermore, the integral can be approximately evaluated for limiting cases of extent in time of $\rho(t_1 - t_2)$ compared to the correlation time of $\mu(t)$ (the width, in time, of the principal maximum of the ambiguity function, $\approx 1/W$).

First, if $\rho(t_1 - t_2)$ is essentially constant over the nonzero interval of $\mu(t)$, as indicated in Fig. 4(a) (corresponding to a very narrow-band distortion), then

$$\sigma_\lambda^2 \approx \sigma_\delta^2 \left|\int_0^T \mu_1^*(t_1)\,\mu_2(t_1)\,dt_1\right|^2$$

$$= \sigma_\delta^2\,|\bar{\lambda}_{12}|^2 \quad,$$

indicating that $\sigma_\lambda \ll \sigma_\delta$ in the ambiguity function side-lobe region. However, $\rho(t_1 - t_2)$ is essentially constant over the entire interval of nonzero $\mu(t)$. Thus, such error represents only a scaling of $\mu(t)$ and, consequently, a scaling of λ_{12}. In effect, it represents a component of unknown phase and amplitude in the return. Since we will later consider signals of unknown phase and amplitude, we will not give further consideration to slowly varying ρ's.

At the other extreme of the distortion picture, let us assume the distortion to be quite wideband; specifically of some distortion bandwidth W_D. If we bear in mind that $\rho(t_1 - t_2)$ is the

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† Also called a "modified Rayleigh" density function by some.

‡ The problem of estimating $\sigma_{\bar{\lambda}}$ is still by no means trivial. A major obstacle is a decision about the shape we should attribute to λ_{12}. For many practical cases, the side lobe structure of the ambiguity function will be almost all "error," in the sense that the structure will bear little resemblance to the "paper design" shape. This suggests that we should set $\lambda_{12} = 0$ throughout the side-lobe region as a practical compromise, and relegate the entire side-lobe structure (averaged over some hopefully representative sampling of targets) to some numerical σ_λ. At the other extreme, we could set aside five minutes of each hour to make repeated measurements of the ambiguity function, its mean and its variance (on some thoughtfully provided representative targets), and use these estimates for the remainder of the hour. The latter procedure would undoubtedly result in a much smaller value being assigned to σ_λ, at a price.

Fourier transform of some spectral density, we can see that the duration over which $\rho(t_1 - t_2)$ is large will be of order $1/W_D$. Thus, for large TW coded pulses and W_D of the order of the modulation bandwidth, the situation will be as indicated in Fig. 4(b). A useful model of $\rho(t_1 - t_2)$ that will aid in visualization (and will be used extensively in later sections) is

$$\rho(t_1 - t_2) = \operatorname{sinc} W_D(t_1 - t_2)$$

where, for brevity, we follow Woodward's[5] notation

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x} \quad .$$

This form of ρ corresponds, of course, to a distortion spectral density of uniform amplitude over a band of width W_D cycles centered on the carrier frequency, and zero everywhere else.

As $\rho(t_1 - t_2)$ becomes very narrow, it acts much like a "delta function" of area $1/W_D$, and Eq. (20) becomes

$$\sigma_\lambda^2 \approx \sigma_\delta^2 \int_0^T |\mu_1(t_1)|^2 \, |\mu_2(t_1)|^2 \int_0^T \rho(t_1 - t_2) \, dt_2 \, dt_1$$

which, for the form of $\rho(t_1 - t_2)$ assumed, for T very large, yields

$$\sigma_\lambda^2 \approx \frac{\sigma_\delta^2}{W_D} \int_0^T |\mu_1(t_1)|^2 \, |\mu_2(t_1)|^2 \, dt_1 \quad . \tag{21}$$

From the normalization constraint, and the assumption of constant signal amplitude, the integral of Eq. (21) is expressible as the ratio T_ℓ/T_o^2, where T_ℓ is the time overlap of Signals 1 and 2 and T_o is the total time the radar transmitter is on for each signal transmission. Thus, for broad-band distortion with distortion bandwidth of the order of W_D,

$$\sigma_\lambda^2 \approx \frac{\sigma_\delta^2}{W_D T_o} \frac{T_\ell}{T_o} \quad . \tag{22}$$

We might physically expect that $W_D T_o$ would approximate the duration-bandwidth product for signals of this type.

It is thus seen that for a <u>fixed</u> value of σ_δ,[*] the larger the duration-bandwidth product, the smaller the effect on the ambiguity function side-lobe structure (a corresponding result for errors in antennas and their patterns is well known).[26]

For a given distortion variance, the variance in the ambiguity function is seen to increase linearly with the degree to which the signals overlap in time for coded-pulse signals. Since it is common practice to concentrate attention on the case of appreciable overlap ($T_\ell \approx T_o$), corresponding to the region of the ambiguity function close to the main lobe in time, we will denote the value of σ_λ^2 for this case by $\sigma_{\lambda_o}^2$, for which we have

$$\sigma_{\lambda_o}^2 = \frac{\sigma_\delta^2}{W_D T_o} \tag{23a}$$

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[*] We might suspect, however, that σ_δ may itself tend to increase with increasing TW.

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and

$$\sigma_\lambda^2 = \sigma_{\lambda_0}^2 \, \frac{T_\ell}{T_0} \quad .$$

(23b)

For pulse-train signals, a form of ρ that seems physically justifiable is one which is wide compared to the width of an individual pulse, but narrow compared to the interpulse spacing, as indicated in Fig. 5. This form is tantamount to assuming essentially constant error over a pulse, and independent errors from pulse to pulse. The constancy of error over the pulse represents a reasonable model for the situation in which the individual pulses are made as short as the radar hardware bandwidths will accommodate (if the radar bandwidths are everywhere wide compared to a reciprocal pulsewidth, each pulse may have wide-band distortion of the type previously treated, but this "combination" case will not be explicitly treated here).

Under these assumptions, t_1 and t_2 of Eq. (20) must differ by less than one pulsewidth or the integral is essentially zero. When $t_1 - t_2$ is thus small, $\rho(t_1 - t_2)$ can be ignored in the integrand, and letting $t_2 = t_1 - \tau$, we can write

$$\sigma_\lambda^2 = \sigma_\delta^2 \int_{-d}^{d} \int_0^{T_0} \mu_1^*(t_1) \, \mu_2(t_1) \, \mu_1(t_1 - \tau) \, \mu_2^*(t_1 - \tau) \, dt_1 \, d\tau \quad ,$$

(24)

where d denotes the pulsewidth of a typical pulse. The offset in the factors of the integrand due to τ just makes the integrand appear to be the convolution of two pulse trains, $\mu_1'(t_1)$ and $\mu_2'(t_1)$, which are related to the original $\mu_1(t_1)$ and $\mu_2(t_1)$ by

$$|\mu_i'(t)| = |\mu_i(t)|^2 \qquad i = 1, 2$$

(recall that a binary amplitude is assumed throughout the train), and the individual pulses of the new train are reduced in width by τ, as indicated in Fig. 6. The value of $|\mu(t)|^2$ is $1/T_0$ by the
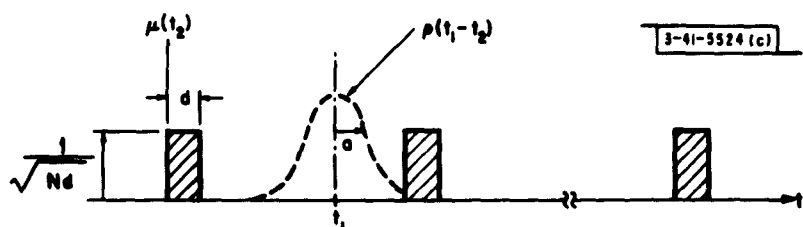

Fig. 5. Assumed scale of the error correlation function ρ, the pulsewidth d and interpulse spacing for pulse to pulse distortion of pulse trains.

Fig. 6. Visualization of products of the form $\mu_1^*(t_1) \, \mu_1(t_1 - \tau)$ of Eq. 24.

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normalization constraint, Eq. (2), where T_o = Nd, the total "on time" of the train. A first integral (on t_1) of Eq. (24) produces a result of the form

$$\sigma_\lambda^2 = \frac{2\sigma_\delta^2}{T_o} \bar{\lambda}_{12} \int_0^d \frac{d - |\tau|}{d} \, d\tau \quad .$$

The result is

$$\sigma_\lambda^2 = \frac{\sigma_\delta^2}{N} \bar{\lambda}_{12} \tag{25}$$

since T_o/d = N, the number of pulses in the train.

A comparison of Eqs. (25) and (22) for the ambiguity function variance for pulse-train signals and continuous signals indicates that for equal σ_δ^2, we have

$$\frac{\sigma_\lambda^2(\text{pulse train})}{\sigma_\lambda^2(\text{coded})} = \frac{\bar{\lambda}_{12}}{N} (W_D T_o) \frac{T_o}{T_l} \quad .$$

The product of distortion bandwidth-time duration for the pulse train can be written as approximately $N\bar{D}/d$ where $\bar{D}$ is the average pulse separation. For equal products, then

$$\frac{\sigma_\lambda^2(\text{pulse train})}{\sigma_\lambda^2(\text{coded})} = \bar{\lambda}_{12} \frac{\bar{D}}{d} \frac{T_o}{T_l} \quad .$$

Where $\bar{\lambda}_{12}$ is nonzero, it is likely to be of the order of 1/N or more;[19] thus, for essentially unity overlap,

$$\frac{\sigma_\lambda^2(\text{pulse train})}{\sigma_\lambda^2(\text{coded})} \approx \frac{\bar{D}}{Nd} \quad .$$

Thus, the ratio may be in favor of either waveform, depending upon the above ratio.

The results of Eqs. (22) and (25) show that when reasonable care in alignment has been taken and small target distortions are present, the percentage effect of the signal distortion on the ambiguity function will be negligible in the main-lobe region of the ambiguity function (where $\bar{\lambda}_{12} \approx 1$), although the errors may completely dominate the side-lobe structure. Thus, the error in the received Signal 2 $[\psi(t, \bar{\gamma}_2)$ of Eq. (11)] will have negligible effect on the ability to detect Signal 2 in the presence of Signal 1.

Consequently, since we will concentrate on detecting Signal 2, it will be assumed that Signal 2 is undistorted for the remainder of the report, but that the interfering Signal 1 is distorted

$$\zeta(t) = v(t) + [\beta_1 + \delta_1(t)] \mu_1(t) + \beta_2 \mu_2(t) \quad . \tag{26}$$

III. THE OPTIMUM RESOLVER

In this chapter, we will derive the optimum circuit upon which to base a decision about our two alternative hypotheses: is one target present or are two? We will analyze the performance of this resolver in terms of the probability that we will correctly say Signal 2 present when it is (the probability of detection P_D), and of the probability that we will say Signal 2 present when it is not (probability of false alarm P_F).

A. Definition of "Optimum"

Before attempting to derive the form of the "optimum" resolver, the sense in which the re-
~olver is to be optimum must be defined. This problem has been discussed at length by
others,[5,11,24] and such discussion will not be repeated in detail here. It will suffice, instead,
to point out that, as one might expect, the most we can ask for from a situation which is basically
probabilistic is further probabilistic information. Consequently, we choose to consider the op-
timum receiver as one which computes for us the probability that one of the hypotheses is true,
given the available data.

The hypothesis we wish to test is that a second signal $\psi_2(t)$ is present, given some received
waveform $\zeta(t)$ known to contain some randomly distorted signal, which we will denote $\psi_1(t)$. In
terms of probability, we wish a circuit which will compute the a posteriori probability
$p[\psi_2(t) | \zeta(t), \psi_1(t)]$.

If $\psi_2(t)$ and $\psi_1(t)$ are uniquely defined signals $[\psi_2(t) = \beta_2\mu(t, \overline{\gamma}_2)$, where β_2 and $\overline{\gamma}_2$ are com-
pletely and uniquely specified], we can make straightforward use of Bayes' Rule[23,24] and write
the a posteriori probability in terms of the a priori probabilities (temporarily suppressing the
time dependence in the interests of more compact notation) as

$$p[\psi_2 | \zeta, \psi_1] = \frac{p[\zeta | \psi_1, \psi_2] \, p[\psi_1, \psi_2]}{p[\zeta | \psi_1, \psi_2] \, p[\psi_1, \psi_2] + p[\zeta | \psi_1] \, p[\psi_1]} \tag{27}$$

where $p[\psi_1]$ denotes the a priori probability of getting the particular, unique $\psi_1(t)$ of concern, and
$p[\psi_1, \psi_2]$ denotes the joint a priori probability of getting both the particular ψ_1 and ψ_2 of concern.

Since any monotonic function of the a posteriori probability is equally optimum, we note that
Eq. (27) is a monotonic function of the "likelihood ratio" $L[\psi_2(t)]$ given by

$$L[\psi_2] = \frac{p[\zeta | \psi_1, \psi_2] \, p[\psi_1, \psi_2]}{p[\zeta | \psi_1] \, p[\psi_1]} \quad .$$

A useful monotonically related function is $\ln L$, particularly when in the form

$$\ln L[\psi_2] = \ln \left[\frac{p[\zeta | \psi_1, \psi_2]}{p[\zeta | \psi_1]} \right] + \ln \left[\frac{p[\psi_1, \psi_2]}{p[\psi_1]} \right] \quad ,$$

since this form separates the a priori probabilities that any targets ψ_1 and ψ_2 are present into a
separate factor which can be considered as part of the threshold setting process on the first term.
We can then confine our attention to the first term, which we will denote $L'[\psi_2(t)]$:

$$L'[\psi_2] = \ln \left[\frac{p[\zeta | \psi_1, \psi_2]}{p[\zeta | \psi_1]} \right] \quad . \tag{28}$$

Our primary interest will be directed toward the situation in which $\psi_1(t)$ and $\psi_2(t)$ are not uniquely
defined, but contain random components (the random distortion process, $\delta(t)$ of Eq. (12); random
complex cross sections, β_i). Equation (28) and its development remain valid for this case if we
slightly reinterpret the symbolism. We are interested in the a posteriori probability of some
class of signals $\psi_2(t)$, given a sample of signal-plus-noise $\zeta(t)$ known to contain one of an en-
semble of possible $\psi_1(t)$'s. If the statistics of the random parameters of $\psi_1(t)$ are known, Eq. (26)
for $\zeta(t)$ allows (at least in principal) the straightforward calculation of $p[\zeta | \psi_1]$ or $p[\zeta | \psi_1, \psi_2]$,
where we now interpret these a priori probabilities as the probability density functions of $\zeta(t)$,
given the statistics of the random parts of $\psi_1(t)$ and/or $\psi_2(t)$.

B. The Optimum Resolver – All Target Parameters Known

In order to first obtain some kind of a solution to the problem of the optimum resolver in a comparatively simple context, let us consider a highly favorable case of a priori knowledge:

$$\psi_1(t) = [\beta_1 + \delta_1(t)]\,\mu(t, \overline{\gamma}_1) \quad ,$$

where β_1, $\overline{\gamma}_1$ and $\mu(t)$ are known exactly, and

$$\psi_2(t) = \beta_2\mu(t, \overline{\gamma}_2)$$

is completely known.[†] Thus, only the distortion of Signal 2 is taken as random. Then $\zeta(t)$, as given by Eq. (26) will be a complex Gaussian random process[6] with a sample probability density function of the form

$$P[\zeta_1, \zeta_2, \ldots, \zeta_N | \psi_1, \psi_2] = \frac{1}{(2\pi)^{N/2}|\Lambda|^{1/2}} \exp\left\{ -\frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} [\zeta_m - \overline{\zeta}_m]\,\Lambda_{mn}^{-1}\,[\zeta_n^* - \overline{\zeta}_n^*] \right\} \tag{29}$$

where ζ_i is the ith sample value of $\zeta(t)$, $\overline{\zeta}_i$ is the average value of that sample, and $[\Lambda]$ is the covariance matrix, a typical element of which is

$$\Lambda_{k\ell} = \overline{(\zeta_k^* - \overline{\zeta}_k^*)(\zeta_\ell - \overline{\zeta}_\ell)} \quad . \tag{30}$$

The symbol $|\Lambda|$ denotes the determinant of the matrix $[\Lambda]$.

Assuming the samples are taken at regular intervals, spaced some Δt apart, we have that for both signals present (the results for Signal 1 only being obtained by setting $\beta_2 = 0$),

$$\overline{\zeta_k} = \beta_1\mu_{1k} + \beta_2\mu_{2k} \tag{30a}$$

and

$$\overline{(\zeta_k^* - \overline{\zeta}_k^*)(\zeta_\ell - \overline{\zeta}_\ell)} = \overline{[\nu_k^* + \delta_{1k}^*\mu_{1k}^*][\nu_\ell + \delta_{1\ell}\mu_{1\ell}]} \tag{30b}$$

where the subscripts k and ℓ denote samples taken at $t = k\Delta t$ and $\ell\Delta t$, respectively. For white noise of spectral density $2N_o$ watts/cycle for positive frequencies, and zero for negative frequencies, the autocorrelation of the noise is[6]

$$\overline{\nu_k^* \nu_\ell} = 2N_o u_o[(k - \ell)\,\Delta t] \tag{31}$$

where $u_o(t)$ denotes the unit impulse function.

We then have, from Eq. (30b),

$$\Lambda_{k\ell} = 2N_o u_o[(k - \ell)\,\Delta t] + 4\sigma_\delta^2 E_1 \rho[(k - \ell)\,\Delta t]\,\mu_{1k}^*\mu_{1\ell}$$

where it is recalled from Eq. (19) that

$$\overline{\delta_{1k}^*\delta_{1\ell}} = 4\sigma_\delta^2 E_1 \rho[(k - \ell)\,\Delta t]$$

[†] Recall that it was shown in Sec. II-B that the effect of the distortion of Signal 2 on the detection of Signal 2 was negligible to a first order.

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where $\rho[(k - \ell)\,\Delta t]$ is the normalized covariance of the $\delta(t)$ process. Writing $R_1 = 2E_1/N_o$, we have

$$\Lambda_{k\ell} = 2N_o \left\{ u_o[(k - \ell)\,\Delta t] + R_1 \sigma_\delta^2 \rho[(k - \ell)\,\Delta t]\,\mu_{1k}^*\mu_{1\ell} \right\} \quad . \tag{32}$$

The problem now becomes one of finding the inverse of the Λ matrix Λ^{-1} such that

$$\sum_{m=1}^{N} \Lambda_{\ell m}\Lambda_{mn}^{-1} = \frac{\sin \pi(\ell - n)}{\pi(\ell - n)} \tag{33}$$

where the use of the $\sin x/x$ function is a convenient method of specifying the value of the sum to be unity for $\ell = n$, and zero otherwise.

To expedite this task,[†] it is useful to make the notational changes

$$\Lambda_{\ell m} = \mathfrak{R}[\ell\Delta t, m\Delta t]$$

$$\Lambda_{mn}^{-1} = r[m\Delta t, n\Delta t]\,(\Delta t)^2$$

and note that (33) can now be written as

$$\sum_{m=1}^{N} \mathfrak{R}[\ell\Delta t, m\Delta t]\,r[m\Delta t, n\Delta t]\,\Delta t = \frac{1}{\Delta t}\,\frac{\sin \frac{\pi[\ell\Delta t - n\Delta t]}{\Delta t}}{\frac{\pi[\ell\Delta t - n\Delta t]}{\Delta t}} \tag{34}$$

If we return to the likelihood ratio of Eq. (28), the ratio of the conditional probabilities of ζ for Gaussian processes is

$$\frac{p[\zeta_1, \ldots, \zeta_N | \psi_1, \psi_2]}{p[\zeta_1, \ldots, \zeta_N | \psi_1]} = \frac{\exp\left\{-\frac{1}{2} \sum_{m,\,n=1}^{N\,N} [\zeta_m - \bar{\zeta}_{m2}]\,\Lambda_{mn2}^{-1}\,[\zeta_n^* - \bar{\zeta}_{n2}^*]\right\}}{\exp\left\{-\frac{1}{2} \sum_{m,\,n=1}^{N} [\zeta_m - \bar{\zeta}_{m1}]\,\Lambda_{mn1}^{-1}\,[\zeta_n^* - \bar{\zeta}_{n1}^*]\right\}}$$

where the subscripts 1 and 2 on the statistics of ζ refer to (1) Signal 1 only, and (2) Signals 1 and 2 present.

If we let $\Delta t \to 0$, $N \to \infty$, such that $N\Delta t = T$, a constant, then we can replace the double sums by integrals, letting $m\Delta t = t_1$, $n\Delta t = t_2$, to yield

$$\frac{p[\zeta(t) | \psi_1(t), \psi_2(t)]}{p[\zeta(t) | \psi_1(t)]}$$

$$= \frac{\exp\left\{-\frac{1}{2} \int\!\!\int_0^T [\zeta(t_1) - \overline{\zeta_2(t_1)}]\,r_2(t_1, t_2)\,[\zeta^*(t_2) - \overline{\zeta_2^*(t_2)}]\,dt_1\,dt_2\right\}}{\exp\left\{-\frac{1}{2} \int\!\!\int_0^T [\zeta(t_1) - \overline{\zeta_1(t_1)}]\,r_1(t_1, t_2)\,[\zeta^*(t_2) - \overline{\zeta_1^*(t_2)}]\,dt_1\,dt_2\right\}} \tag{35}$$

where $r(t_1, t_2)$ satisfies Eq. (34) in the limit. To find this limit, we note that the area under the right side of Eq. (34) is unity, regardless of Δt, while its amplitude approaches infinity at

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[†] The procedure that follows is based upon a suggestion by W.M. Siebert.

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$l\Delta t = n\Delta t$, and zero elsewhere at $\Delta t \to 0$. This suggests that we can write that, in the limit, $r(t_2, t_3)$ must satisfy the integral equation

$$\int_0^T \mathcal{R}(t_1, t_2) \, r(t_2, t_3) \, dt_2 = u_0(t_1 - t_3) \quad . \tag{36}$$

For $\mathcal{R}(t_1, t_2)$, we need only make notational changes in Eq. (32) to establish that

$$\mathcal{R}(t_1, t_2) = 2N_0[u_0(t_1 - t_2) + R_1 \sigma_\delta^2 \rho(t_1 - t_2) \mu_1^*(t_1) \mu_1(t_2)] \quad .$$

Let us assume an $r(t_2, t_3)$ of the form

$$r(t_2, t_3) = \frac{1}{2N_0} [u_0(t_2 - t_3) + A\rho(t_2 - t_3) \mu_1^*(t_2) \mu_1(t_3)]$$

where A is assumed to be a constant that will be determined. We will then investigate the circumstances under which this assumption is at least approximately satisfied.

Substitution into Eq. (36) shows that in order for $r(t_2, t_3)$ to be so represented, we must have

$$R_1 \sigma_\delta^2 \rho(t_1 - t_3) \mu_1^*(t_1) \mu_1(t_3) + A\rho(t_1 - t_3) \mu_1^*(t_1) \mu_1(t_3)$$

$$+ AR_1 \sigma_\delta^2 \mu_1^*(t_1) \mu_1(t_3) \int_0^T |\mu_1(t_2)|^2 \rho(t_1 - t_2) \, \rho(t_2 - t_3) \, dt_2 = 0 \quad . \tag{37}$$

The value of the integral will be considered separately for two different classes of large TW signals: (1) coded pulses, and (2) pulse trains.

For coded pulses, assuming the distortion to be wide-band compared to the reciprocal of the signal duration, the integrand is of the nature indicated in Fig. 7. It is apparent that for very
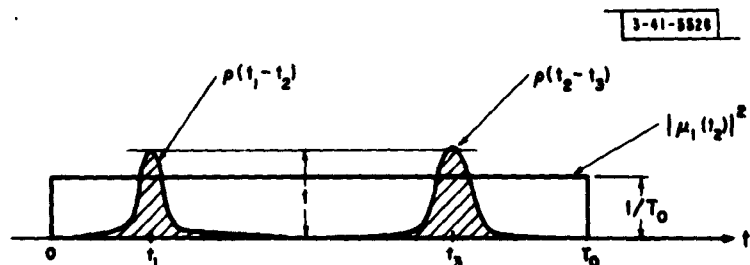

Fig. 7. The structure of $|\mu_1(t_2)|^2 \rho(t_1 - t_2) \, \rho(t_2 - t_3)$ for wide-band distortion of a pulse with continuous rectangular envelope.

large duration-bandwidth signals, the term $|\mu_1(t_2)|^2$ will affect the value of the integral appreciably only when t_1 and/or t_3 are near the edges of the pulse. If we consider the pulse to last from $t = 0$ to $t = T_0$, and note that from the normalization constraint $|\mu_1(t)|^2 = 1/T_0$, we can write the integral as

$$\int_0^T |\mu_1(t_2)|^2 \rho(t_1 - t_2) \, \rho(t_2 - t_3) \, dt_2 \approx \frac{1}{T_0} \int_0^{T_0} \rho(t_1 - t_2) \, \rho(t_2 - t_3) \, dt_2 \quad .$$

If t_2 and/or t_3 is outside the interval 0 to T_o, $\mu_1^*(t_1)$ and/or $\mu_1(t_3)$ will be zero and the equation is satisfied, regardless of the integral. Consequently, if the distortion is truly wide-band, compared to $1/T_o$, the limits on the integral can go from $-\infty$ to ∞ with little error (edge effect), and we then are seeking a solution to

$$[R_1\sigma_\delta^2 + A]\,\rho(t_1 - t_3) + \frac{A}{T_o}\,R_1\sigma_\delta^2 \int_{-\infty}^{\infty} \rho(t_1 - t_2)\,\rho(t_2 - t_3)\,dt_2 = 0 \quad,$$

$$0 < t_1 < T_o, \ \ 0 < t_3 < T_o \quad . \qquad\qquad (38)$$

It is obvious that this equation can be satisfied if ρ is of such a form that, when convolved with itself, the result is again ρ, to within a scale factor. Such a function is the "sinc" function introduced in Sec. II-D:

$$\rho(t_1 - t_3) = \text{sinc}\,W_D(t_1 - t_3) \quad .$$

This choice of $\rho(t_1 - t_3)$ implies that the spectral density of $\delta(t)$ is uniform over some distortion bandwidth W_D, a not unreasonable model. With this form of ρ, Eq. (38) becomes

$$R_1\sigma_\delta^2 + A + \frac{A}{W_D T_o}\,R_1\sigma_\delta^2 = 0$$

or, from Eq. (23), relating σ_δ^2 and σ_λ^2,

$$A = \frac{-R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2} \ .$$

Thus, for a signal with a rectangular envelope of duration T_o and a random distortion that has a flat spectral density over a bandwidth $W_D \gg 1/T_o$,

$$r(t_1, t_2) \approx \frac{1}{2N_o}\left[u_o(t_1 - t_2) - \frac{R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2}\,\mu_1^*(t_1)\,\mu_1(t_2)\,\text{sinc}\,W_D(t_1 - t_2)\right] \quad . \qquad (39)$$

Combining this result for both $r_1(t_1, t_2)$ and $r_2(t_1, t_2)$ of Eq. (35) with Eq. (30a) gives the form of the optimum resolver as one which computes

$$L''[\psi_2(t)] = \text{Re}\,\beta_2^*\left[\int_0^T \zeta(t)\,\mu_2^*(t)\,dt\right.$$

$$\left. - \frac{R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2}\iint_0^T \zeta(t_1)\,\mu_1^*(t_1)\,\text{sinc}\,W_D(t_1 - t_2)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2\right] \qquad (40)$$

where $L''[\psi_2(t)]$ differs from $L'[\psi_2(t)]$ of Eq. (28) only by a factor which does not involve the received data of $\zeta(t)$, and therefore is absorbed into the question of threshold settings. Discussion of the significance of Eq. (40) will be postponed until the optimum resolver for pulse-train signals is derived.

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Let us now return to Eq. (37) and pursue a solution for pulse-train waveforms of the type shown in Figs. 1(c) and 1(d). We will assume that the individual pulses are not coded, and that the errors are highly correlated over a single pulse, but essentially uncorrelated pulse to pulse. A convenient model of $\rho(t_1 - t_2)$ that meets these requirements is a Gaussian shape

$$\rho(t_1 - t_2) = \exp\left[-\frac{(t_1 - t_2)^2}{2a^2}\right]$$

as indicated in Fig. 5, where a is assumed greater than the pulsewidth d but much smaller than the smallest interpulse period D_n. Then Eq. (37) is approximately

$$[R_1 \sigma_\delta^2 + A] \exp\left[-\frac{(t_1 - t_3)^2}{2a^2}\right]$$

$$+ AR_1 \sigma_\delta^2 \exp\left[-\frac{(t_1 - t_3)^2}{4a^2}\right] \int_0^T |\mu_1(t_2)|^2 \exp\left\{-\frac{1}{a^2}\left[t_2 - \frac{t_1 + t_3}{2}\right]^2\right\} dt_2 = 0$$

for t_1 and t_3 such that $\mu_1^*(t_1)\,\mu_1(t_3) \neq 0$.

Under the above assumptions, the Gaussians involving t_1 and t_3 are nearly unity when t_1 and t_3 differ by less than a pulsewidth, and essentially zero for t_1 and t_3 separated by an interpulse period or greater. Therefore, for $t_1 \approx t_3$,

$$R_1 \sigma_\delta^2 + A + AR_1 \frac{\sigma_\delta^2}{N} = 0$$

since the last integral is approximately the ratio of the area under a typical pulse to the area under the entire train $1/N$. Thus, for pulse trains of equal-width pulses

$$A = \frac{-R_1 \sigma_\delta^2}{1 + \frac{R_1}{N}\sigma_\delta^2} \ . \tag{41}$$

Recalling from Sec. II-D that we can establish a correspondence between the number of pulses in a train N and the distortion-bandwidth and time-duration product $W_D T_o$, comparison of this result with the previous value of A for coded pulses indicates that within the accuracy of the respective approximations, the results are essentially identical.

With this identification, Eq. (40) becomes the form of the optimum detector for both pulse-train signals and those with continuous rectangular envelopes, except for the differing forms of $\rho(t_1 - t_2)$ assumed, and the differing constants multiplying the double integral.

C. A Physical Interpretation of the Optimum Resolver

Inspection of Eq. (40) shows that the resolver consists of two parts. The first single integral is identifiable as the output at time zero of a "matched" filter with impulse response

$$h(t) = \beta_2^* \mu_2^*(-t)\, e^{+j\omega_o t} \ .$$

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Identifying the double integral is more difficult. However, if we write the integral as

$$\beta_2^* \int_0^T \zeta(t_1) \left[\int_0^T \mu_1^*(t_1) \, \rho(t_1 - t_2) \, \mu_1(t_2) \, \mu_2^*(t_2) \, dt_2 \right] dt_1$$

(where we revert to the general representation for the normalized autocorrelation of the errors to make the results applicable to both types of waveforms under consideration), we note that the term in brackets is just some function of t_1 for specified signals and autocorrelation function.

For example, for long coded pulses and wide-band distortion, the "sinc" assumed for $\rho(t_1 - t_2)$ acts approximately as a delta function of area $1/W_D$, making the term in brackets

$$\beta_2^* \int_0^T \mu_1^*(t_1) \, \rho(t_1 - t_2) \, \mu_1(t_2) \, \mu_2^*(t_2) \, dt_2 \approx \frac{\beta_2^*}{W_D} \, |\mu_1(t_1)|^2 \mu_2^*(t_1)$$

for which the entire double integral can be interpreted as the output at time zero of a filter with impulse response

$$h(t) = \begin{cases} \dfrac{1}{W_D T_o} \beta_2^* \mu_2^*(-t) \, e^{j\omega_o t} & , \quad \text{for } t \text{ such that } |\mu_1(t)|^2 \neq 0 \\[4mm] 0 & , \quad \text{elsewhere} \quad . \end{cases}$$

Thus, the optimum filter for this case is one with an impulse response

$$\beta_2^* \mu_2^*(-t) \left[1 - \frac{R_1 \sigma_{\lambda_o}^2}{1 + R_1 \sigma_{\lambda_o}^2} \text{rect} \, [\mu_1(-t)] \right] \exp \, [j\omega_o t]$$

where we define the function "$\text{rect} \, \mu_1(t)$" as unity during the duration of $|\mu_1(t)|$, and as zero elsewhere. The detector is thus of the form of Fig. 8 for this type of signal and distortion.
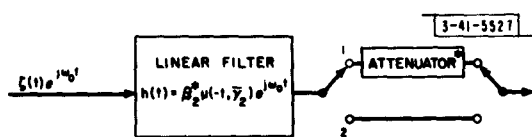

Fig. 8. Optimum resolver for coded pulse signals with wide-band distortion, all parameters of both signals known exactly. Switch in Position 1 during time $\mu(t, \bar{\gamma}_1)$ is nonzero. Attenuation by factor $1/(1 + R_1 \sigma_{\lambda_o}^2)$.

During the portion of $\mu_2(t)$ which μ_1 does not overlap, we convolve with $\beta_2^* \mu_2(t) \exp \, [j\omega_o t]$ as we would if Signal 1 were never expected. During the overlap, we weight the filter by the factor

$$\frac{1}{1 + R_1 \sigma_{\lambda_o}^2}$$

which is seen to have a value of unity if $R_1 \sigma_{\lambda_o}^2$ is small. Note that $R_1 \sigma_{\lambda_o}^2$ is approximately the ratio of the mean-square interference of Signal 1 at Signal 2's location to the noise power density, if the overlap is very large.

D. The Performance of the Optimum Resolver – All Target Parameters Known

In this section, we will derive performance curves for the optimum resolver of Eq. (40) in terms of the probability of detection P_D and probability of false alarm P_F for a Signal 2 in the presence of Signal 1.

We note from Fig. 8 that the resolver is a linear device (although perhaps time varying) so that the output will be a Gaussian process, as is the input process. Thus, we need compute only the mean and variance of the output.

We assume that the input is of the form of Eq. (26). Then, from Eq. (40),

$$
\begin{aligned}
L''[\psi_2(t)] = \mathrm{Re} \Bigg\{ & \beta_2^* \int_0^T \upsilon(t)\, \mu_2^*(t)\, dt + \beta_1 \beta_2^* \overline{\lambda}_{12} \\
& + \beta_2^* \int_0^T \delta_1(t)\, \mu_1(t)\, \mu_2^*(t)\, dt + |\beta_2|^2 \\
& - \frac{R_1 \sigma_\delta^2}{1 + R_1 \sigma_{\lambda_0}^2} \Bigg[\beta_2^* \iint_0^T \upsilon(t_1)\, \mu_1^*(t_1)\, \rho(t_1 - t_2)\, \mu_1(t_2)\, \mu_2^*(t_2)\, dt_1\, dt_2 \\
& + \beta_1 \beta_2^* \iint_0^T |\mu_1(t_1)|^2\, \rho(t_1 - t_2)\, \mu_1(t_2)\, \mu_2^*(t_2)\, dt_1\, dt_2 \\
& + \beta_2^* \iint_0^T \delta_1(t_1)\, |\mu_1(t_1)|^2\, \rho(t_1 - t_2)\, \mu_1(t_2)\, \mu_2^*(t_2)\, dt_1\, dt_2 \\
& + |\beta_2|^2 \iint_0^T \mu_2(t_1)\, \mu_1^*(t_1)\, \rho(t_1 - t_2)\, \mu_1(t_2)\, \mu_2^*(t_2)\, dt_1\, dt_2 \Bigg] \Bigg\}
\end{aligned} \tag{42}
$$

when $\rho(t_1 - t_2) = \mathrm{sinc}\, W_D(t_1 - t_2)$. If Signal 2 is present the mean of L'' is

$$
\begin{aligned}
\overline{L_2''} = \mathrm{Re} \Bigg\{ & \beta_1 \beta_2^* \overline{\lambda}_{12} + |\beta_2|^2 \\
& - \frac{R_1 \sigma_\delta^2}{1 + R_1 \sigma_{\lambda_0}^2} \Bigg[\beta_1 \beta_2^* \iint_0^T |\mu_1(t_1)|^2\, \rho(t_1 - t_2)\, \mu_1(t_2)\, \mu_2^*(t_2)\, dt_1\, dt_2 + |\beta_2|^2 \frac{\sigma_\lambda^2}{\sigma_\delta^2} \Bigg] \Bigg\}
\end{aligned} \tag{43}
$$

where we have made use of Eq. (20) in the last term. When Signal 1 only is present, the mean is given by setting $\beta_2 = 0$ (but note that β_2^* arises from the form of the resolver and is not set to zero), giving

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$$\overline{L_1''} = \mathrm{Re}\,\beta_1\beta_2^* \left[\overline{\lambda}_{12} - \frac{R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2} \int_0^T\!\!\int |\mu_1(t_1)|^2\,\rho(t_1 - t_2)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2 \right] \quad . \tag{44}$$

Thus, the presence of Signal 2 changes the mean of the output by

$$\overline{\Delta L''} = |\beta_2|^2 \frac{1 + R_1\sigma_{\lambda_o}^2 \dfrac{(T_o - T_\ell)}{T_o}}{1 + R_1\sigma_{\lambda_o}^2} \tag{45a}$$

where we have let $\rho(t_1 - t_2) \doteq \mathrm{sinc}\,W_D(t_1 - t_2)$ and assumed "edge effects" negligible. Reference to Eqs. (41) and (25) indicates that the form of (45a) for wide-band distortion of pulse-train signals is

$$\overline{\Delta L''} = |\beta_2|^2 \frac{1 + R_1\sigma_\lambda^2 \left(\dfrac{1 - \overline{\lambda}_{12}}{\overline{\lambda}_{12}}\right)}{1 + R_1 \dfrac{\sigma_\lambda^2}{\overline{\lambda}_{12}}} \quad . \tag{45b}$$

We see from the above results that, for either Signal 2 present or absent, in notation appropriate for coded pulses,

$$L'' - \overline{L''} = \mathrm{Re}\,\beta_2^* \left\{ \int_0^T [\upsilon(t) + \delta_1(t)\,\mu_1(t)]\,\mu_2^*(t)\,dt \right.$$

$$\left. - \frac{R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2} \int_0^T\!\!\int [\upsilon(t_1) + \delta_1(t_1)\,\mu_1(t_1)]\,\mu_1^*(t_1)\,\rho(t_1 - t_2)\,\mu_1(t_2)\,\mu_2^*(t_2)\,dt_1\,dt_2 \right\} \tag{46}$$

from which the variance of L can be calculated, noting that the variance of the real part of the expression is half the variance of the complex process:

$$\sigma_L^2 = \overline{(L'' - \overline{L''})^2} = |\beta_2|^2\,N_o \left\{ 1 + R_1\sigma_\lambda^2 - \frac{2R_1\sigma_\delta^2}{1 + R_1\sigma_{\lambda_o}^2} \left[\frac{\sigma_\lambda^2}{\sigma_\delta^2} + R_1\sigma_\delta^2 \int_0^T\!\!\!\int\!\!\int |\mu_1(t_2)|^2\,\mu_2^*(t_1)\,\mu_1(t_1) \right.\right.$$

$$\left. \times \rho(t_1 - t_2)\,\rho^*(t_2 - t_3)\,\mu_1^*(t_3)\,\mu_2(t_3)\,dt_1\,dt_2\,dt_3 \right] + \frac{(R_1\sigma_\delta^2)^2}{\left(1 + R_1\sigma_{\lambda_o}^2\right)^2} \left[\int_0^T\!\!\!\int\!\!\int |\mu_1(t_1)|^2 \right.$$

$$\times \mu_1(t_2)\,\mu_2^*(t_2)\,\rho(t_1 - t_2)\,\rho^*(t_1 - t_3)\,\mu_1^*(t_3)\,\mu_2(t_3)\,dt_1\,dt_2\,dt_3 + R_1\sigma_\delta^2 \int_0^T\!\!\!\int\!\!\!\int\!\!\int |\mu_1(t_1)|^2$$

$$\left.\left. \times \mu_1(t_2)\,\mu_2^*(t_2)\,\rho(t_1 - t_2)\,\rho(t_1 - t_3)\,\rho^*(t_3 - t_4)|\mu_1(t_3)|^2\,\mu_1^*(t_4)\,\mu_2(t_4)\,dt_1\,dt_2\,dt_3\,dt_4 \right]\right\}$$
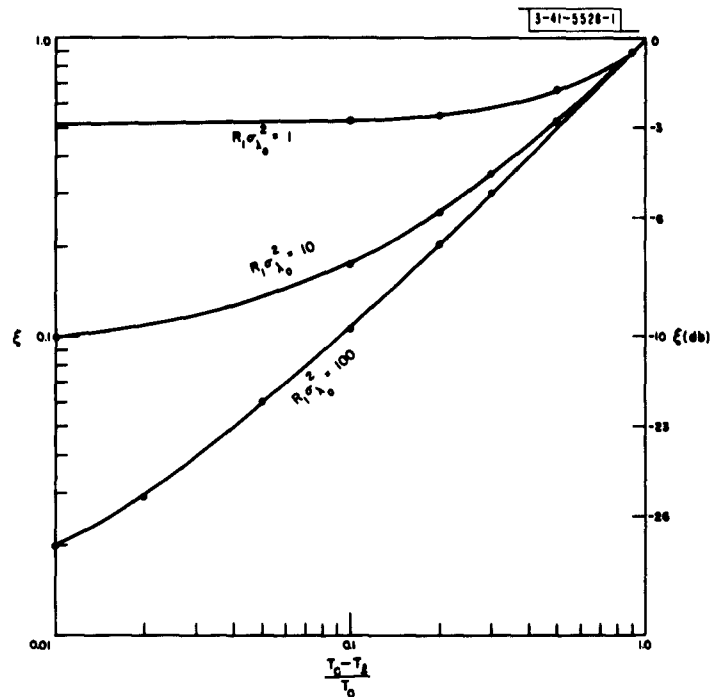
Fig. 9. Plot of overlap factor ξ vs overlap in received signals with interference-to-noise ratio $R_1 \sigma_{\lambda_0}^2$ as a parameter for coded waveforms $R_1 = 2E_1/N_0$ = ratio of energy in signal due to noise power density. $\sigma_{\lambda_0}^2$ = variance in ambiguity function due to waveform distortion in region near main lobe.
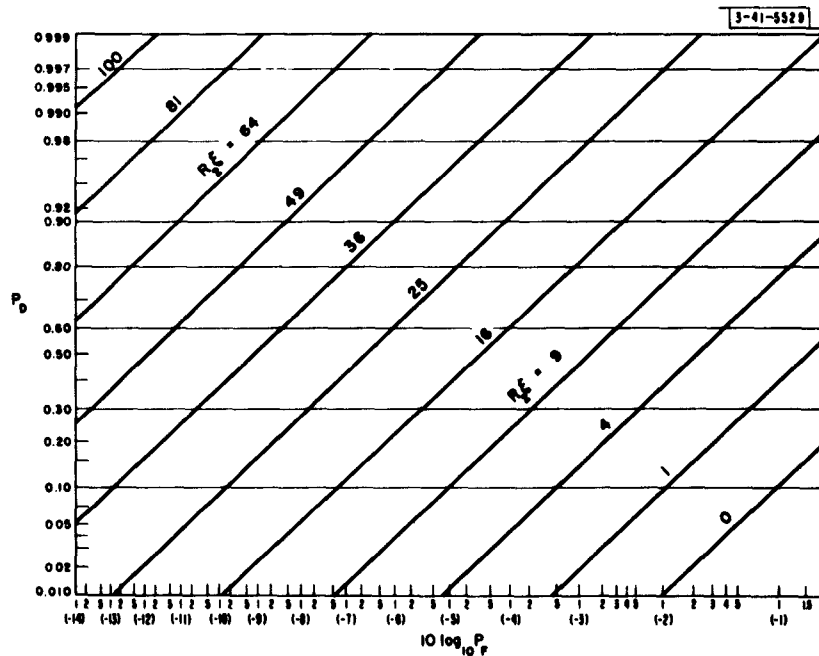
Fig. 10. Probability of detection vs probability of false alarm for second signal overlapped by first signal with overlap parameter ξ as given by Fig. 9.

where we have used Eqs. (19) and (20).

For the long, coded pulse, setting $\rho(t_i - t_j) = \operatorname{sinc} W_D(t_i - t_j)$, assuming wide-band distortion, and neglecting end effects, gives after some algebra,

$$\sigma_L^2 = |\beta_2|^2 \, N_0 \frac{1 + R_1 \sigma_{\lambda_0}^2 \left(\dfrac{T_0 - T_\ell}{T_0}\right)}{1 + R_1 \sigma_{\lambda_0}^2} \quad . \tag{47}$$

Thus, from Eq. (45) and Eq. (47), we see that successful detection involves choosing correctly from two Gaussian distributions of differing means and equal variances. The ratio of the square of the difference in means to the variance is

$$\frac{(\Delta\bar{L})^2}{\sigma_L^2} = R_2 \frac{1 + R_1 \sigma_{\lambda_0}^2 \left(\dfrac{T_0 - T_\ell}{T_0}\right)}{1 + R_1 \sigma_{\lambda_0}^2} \quad . \tag{48}$$

A similar analysis for pulse-train signals with errors correlated during the pulse and decorrelated pulse to pulse, results in

$$\frac{(\Delta\bar{L})^2}{\sigma_L^2} = R_2 \frac{1 + R_1 \sigma_\lambda^2 \left(\dfrac{1 - \bar{\lambda}_{12}}{\bar{\lambda}_{12}}\right)}{1 + R_1 \dfrac{\sigma_\lambda^2}{\bar{\lambda}_{12}}} \quad . \tag{49}$$

Note that both results can be written in the form

$$\frac{(\Delta\bar{L})^2}{\sigma_L^2} = R_2 \, \xi \quad ,$$

where the quantity ξ can be interpreted as an "overlap factor" and varies both with the degree of time overlap T_ℓ/T_0 for long duration-coded pulses, and with the "no-error" ambiguity function $\bar{\lambda}_{12}$ for pulse trains. The overlap factor is plotted in Fig. 9 for coded waveforms.

The significance of this parameter can be appreciated from P_D vs P_F considerations. To compute the performance curves, we need only note that for $T_\ell = 0$, we have the classic case for detection of a single completely known signal,[5,11,24,27] and the $P_D - P_F$ curves for that case are applicable,* with only the values relabeled. Figure 10 is a set of such curves from Manasse,[27] appropriately relabeled.

Note that, in Eqs. (48) and (49), we can set

$$R_1 = rR_2 \quad ,$$

where r is the ratio of the radar cross section of Target 1 to Target 2. Specializing our attention to Eq. (48) for coded waveforms, note that if $R_1 \sigma_{\lambda_0}^2 \ll 1$, the problem is essentially the

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* Note that, for a single known signal, the mean is zero when no signal is present, whereas Eq. (44) indicates that we must account for a mean which involves λ_{12} in the threshold setting when we seek a second target overlapped by a first.

detection of R_2 in a noise-only background. Consequently, we are interested here in the case $R_1 \sigma_{\lambda_0}^2 \gg 1$, and we can neglect unity in the denominator. The effect of the interference can then be assessed by setting the right side of Eq. (48) equal to the value of $R_2 \xi$ desired which we will denote R_0 (note that for reliable detection R_0 is probably on the order of 50). Solving for the value R_2 gives

$$R_2 = \frac{R_0 \, r\sigma_{\lambda_0}^2 - 1}{r\sigma_{\lambda_0}^2 \, \dfrac{T_0 - T_\ell}{T_0}} \quad .$$

It is apparent that if the product of the cross-section ratio and the "close-in" ambiguity function error variance is much greater than $1/R_0$ (a side lobe uncertainty of -30 db and a cross-section difference of 20 db, if $R_0 = 50$), we can neglect unity in the numerator and write

$$R_2 \approx \frac{R_0}{\dfrac{T_0 - T_\ell}{T_0}} \quad , \qquad r\sigma_{\lambda_0}^2 \gg \frac{1}{R_0} \quad . \tag{50}$$

E. The Optimum Resolver – Phase and Amplitude of Signals Unknown – All Other Parameters Known

In order to qualitatively estimate the effect of our assumption of complete knowledge of both signals, we will explore in this section the effect produced by lack of knowledge of either the phase or the amplitude of the return from the target; first, the case of interfering target cross section only unknown, and then both cross sections unknown. It is assumed that the target cross sections are samples of complex Gaussian populations. The two cross sections of interest, β_1 and β_2, are further assumed to be independent and of mean zero and to allow the possibility that they arise from separate populations, we will assume that $|\beta_1|^2 = 2E_{10}$ and $|\beta_2|^2 = 2E_{20}$ (note that we will henceforth distinguish between the actual energy in the real part of a particular return E_i and its statistical average E_{i0}). This model corresponds physically to a Rayleigh amplitude distribution and a uniform phase distribution.

Since the similarity of results for both coded-pulse and pulse-train signals has been demonstrated, we will confine attention to the former in this section.

The wide-band distortion process will be assumed to be a complex Gaussian process as before. However, in order to solve the problem with any degree of facility, it will be necessary to assume β_1 and $\delta_1(t)$ to be independent, except that we may choose the variance of the δ-process to be related to the variance of β_1 in a manner analogous to Eq. (19):

$$\overline{|\delta_1(t)|^2} = 4E_{10}\sigma_\delta^2 \quad . \tag{51}$$

While this assumption is made primarily as a mathematical convenience, it would seem to represent a "worst case," in that any known correlation could be capitalized upon to our advantage. Thus, the result will be meaningful – at least as a limiting case.

The simplest approach to this problem is to consider β_2 fixed at first, so that [compare with Eqs. (30a) and (30b)]

$$\overline{\zeta(t)} = \beta_2 \mu_2(t)$$

and

$$\mathcal{R}(t_1 - t_2) = \overline{[\zeta^*(t_1) - \overline{\zeta^*(t_1)}]} \, \overline{[\zeta(t_2) - \overline{\zeta(t_2)}]}$$

$$= 2N_0 \left\{ u_0(t_1 - t_2) + \frac{R_{10}}{2} \, [1 + 2\sigma_\delta^2 \, \rho(t_1 - t_2)] \, \mu_1^*(t_1) \, \mu_1(t_2) \right\} \quad , \quad (52)$$

where $R_{10} = 2E_{10}/N_0$. If we assume $\rho(t_1 - t_2)$ of the form sinc $W_D(t_1 - t_2)$ and ignore end effects, it is found that

$$r(t_1 - t_2) = \frac{1}{2N_0} \left\{ u_0(t_1 - t_2) - \frac{[1 + R_{10}\sigma_\delta^2 \, \rho(t_1 - t_2)]}{1 + R_{10}\sigma_{\lambda_0}^2} \, \mu_1^*(t_1) \, \mu_1(t_2) \right\} \quad . \quad (53)$$

If we now form the logarithm of likelihood ratio for a fixed β_2, we find that the terms involving the received data are, after some algebra,

$$L''[\psi_2] = \mathrm{Re}\, \beta_2^* \left[\int_0^T \zeta(t) \, \mu_2^*(t) \, dt - \frac{\overline{\lambda}_{12}}{1 + R_{10}\sigma_{\lambda_0}^2} \int_0^T \zeta(t) \, \mu_1^*(t) \, dt \right.$$

$$\left. - \frac{R_{10}\sigma_\delta^2}{1 + R_{10}\sigma_{\lambda_0}^2} \int\!\!\!\int_0^T \zeta(t_1) \, \mu_1^*(t_1) \, \rho(t_1 - t_2) \, \mu_1(t_2) \, \mu_2^*(t_2) \, dt_2 \right] \quad . \quad (54)$$

A schematic of such a detector is indicated in Fig. 11. Comparison of this result with the optimum detector of Eq. (40) (for all parameters known) indicates that for targets separated so
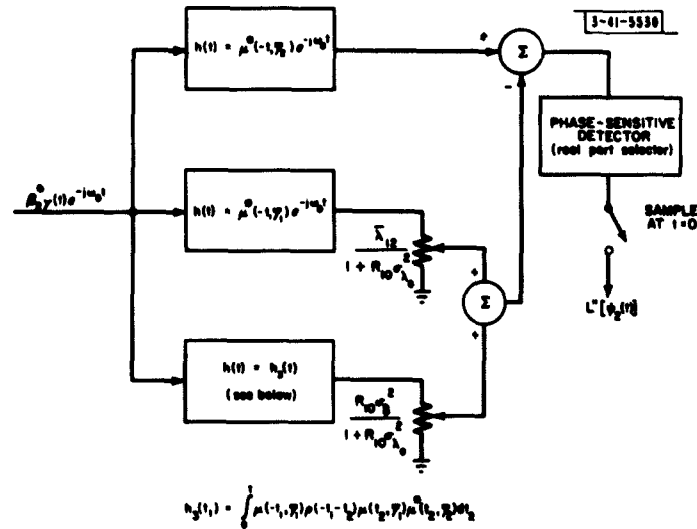

$$h_3(t_1) = \int_0^1 \mu(-t_1, \overline{\tau}) \rho(-t_1 - t) \mu(t_2, \overline{\tau}) \mu^*(t_2, \overline{\tau}) dt_2$$

Fig. 11. Schematic of optimum resolver of Eq. (54) for β_1 unknown in phase and amplitude.

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that $\overline{|\lambda_{12}|} \ll 1$, the results approach equality, except that the quantity R_1 of Eq. (40) is replaced by its expected value R_{10} in Eq. (54). In any event, the resolver of Eq. (54) is similar in structure to that of Eq. (40) (Fig. 8), except for one additional "box": a weighted filter matched to the undistorted Signal 1.

If the statistics of L" are examined, straightforward mathematics will prove that

$$\overline{L"}\big|_{\beta_2=0} = 0 \quad , \tag{55}$$

$$\overline{L"}\big|_{\beta_2\neq0} = \overline{\Delta L} = |\beta_2|^2 \frac{1 - |\overline{\lambda}_{12}|^2 + R_{10}\sigma_{\lambda_0}^2 \left(\frac{T_0 - T_\ell}{T_0}\right)}{1 + R_{10}\sigma_{\lambda_0}^2} \quad , \tag{56}$$

and

$$\sigma_L^2 = |\beta_2|^2 N_0 \frac{1 - \overline{|\lambda_{12}|}^2 + R_{10}\sigma_{\lambda_0}^2 \left(\frac{T_0 - T_\ell}{T_0}\right)}{1 + R_{10}\sigma_{\lambda_0}^2} \quad . \tag{57}$$

The detection parameter of interest becomes

$$\frac{(\overline{\Delta L"})^2}{\sigma_L^2} = R_2 \frac{1 - \overline{|\lambda_{12}|}^2 + R_{10}\sigma_{\lambda_0}^2 \left(\frac{T_0 - T_\ell}{T_0}\right)}{1 + R_{10}\sigma_{\lambda_0}^2} \tag{58}$$

[compare with Eq. (48)].

Thus, the resolver performance is given by the curves of Fig. 9, with ξ replaced by

$$\xi' = \frac{1 - \overline{|\lambda_{12}|}^2 + \sigma_{\lambda_0}^2 \left(\frac{T_0 - T_\ell}{T_0}\right)}{1 + R_{10}\sigma_{\lambda_0}^2} \quad . \tag{59}$$

Note that our lack of knowledge of Signal 1 has indeed cost us something. For the first time, even in the absence of errors, we must pay for $|\lambda_{12}|$ being large, which agrees qualitatively with Helstrom's results[3] (see Sec. I-A) for signals with unknown parameters but known λ's. Of course, if $|\lambda_{12}| \ll 1$, as would normally be true if the targets are separated by the width of the main lobe of the ambiguity function, the result is essentially of the same form as for Signal 1 known completely, except that R_1 is replaced by its average R_{10}.

In order to make β_2 appear random, we need only multiply the antilog of Eq. (54) by the a priori distribution of β_2 and average over all β_2

$$L[\psi_2, E_{20}] = \int_{-\infty}^{\infty} \exp\left\{-\frac{L"[\psi_2]}{N_0}\right\} \exp\left\{-\frac{|\beta_2|^2}{2E_{20}}\right\} d\beta_2 \quad .$$

The result is the same as the corresponding exercise for single-target detection.[5,11,24,27] That is, we should envelope-detect $L''(\psi_2)$ of Eq. (54). The effect on detectability is well known.[5] For a given $|\beta_2|$, the curves of Fig. 10 are displaced[†] to form those of Fig. 12 (see Ref. 27) where ξ' is given by Eq. (59). It is apparent that for a given cross-section magnitude, the additional price which we must pay for confessing our ignorance of exact cross section in the resolver design is quite small at levels of high P_D and small P_F.

Thus, it is apparent that for large overlap $[T_\ell/T_O \approx 1]$, resolution with high probability is nearly impossible if the ambiguity function cannot be controlled so that $r\sigma_{\lambda_O}^2$ is very small. However, for this optimum resolver, the situation improves rapidly to the point that for as much as 90 percent overlap $[(T_O - T_\ell)/T_O = 0.1]$, we pay only a 10-db penalty.

A comparison of Eqs. (48) and (49) indicates that if we can realize both a pulse train and a long, rectangular envelope signal with the same σ_λ^2, since the parameter $(1 - \bar{\lambda}_{12})$ for the pulse train can be made to drop to a small value in a time about equal to the width of one pulse of the train, the pulse train might provide superior resolution. As pointed out in Sec. II-D, however, the relative value of σ_λ^2 that can be expected for equal care in hardware design and alignment with the two signal types, is not clear at this time.

IV. THE PERFORMANCE OF SOME NONOPTIMUM RESOLVERS

This study would be incomplete if we did not conclude with an excursion into the question of "How optimum is optimum?" "Are these optimum resolvers sufficiently superior to some simpler, nonoptimum devices to be worthy of practical consideration?" In this section, we will attempt to at least partially answer the question with some examples.

A. The Performance of an Optimum "No-Error" Resolver in the Presence of Error

A direction which suggests itself for an initial venture away from the optimum is to ignore the error in the signal structure in the design of the receiver. Specifically, let us assume that the problem is that of the previous section: resolution of targets of unknown complex cross sections, but all other parameters known.

Instead of constructing exactly the filter of Eq. (54) (Fig. 11), let us examine the effect of assuming $\sigma_{\lambda_O}^2 \equiv 0$, and of constructing a filter $F_1[\psi_2]$, such that

$$F_1[\psi_2(t)] = |F_1'[\psi_2(t)]| \quad ,$$

where

$$F_1'[\psi_2(t)] = \int_0^T \zeta(t) \, [\mu_2^*(t) - \bar{\lambda}_{12}\mu_1^*(t)] \, dt \quad . \tag{60}$$

Then, for

$$\zeta(t) = \nu(t) + [\beta_1 + \delta_1(t)]\mu_1(t) + \beta_2\mu_2(t) \quad ,$$

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† Note that a different set of curves[11] is generated if we inquire into P_D vs P_F for any member of the population of cross sections characterized by the assumed distribution for β_2.
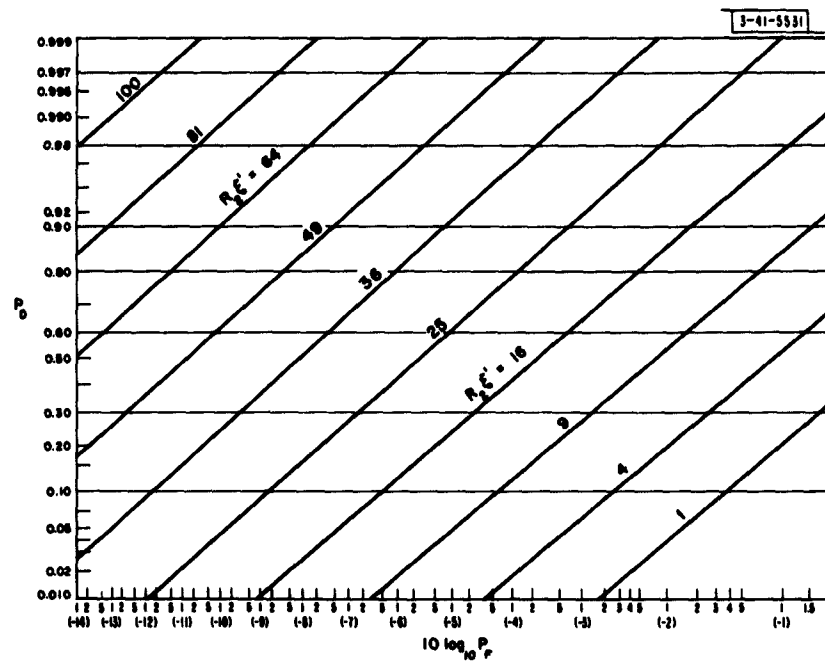
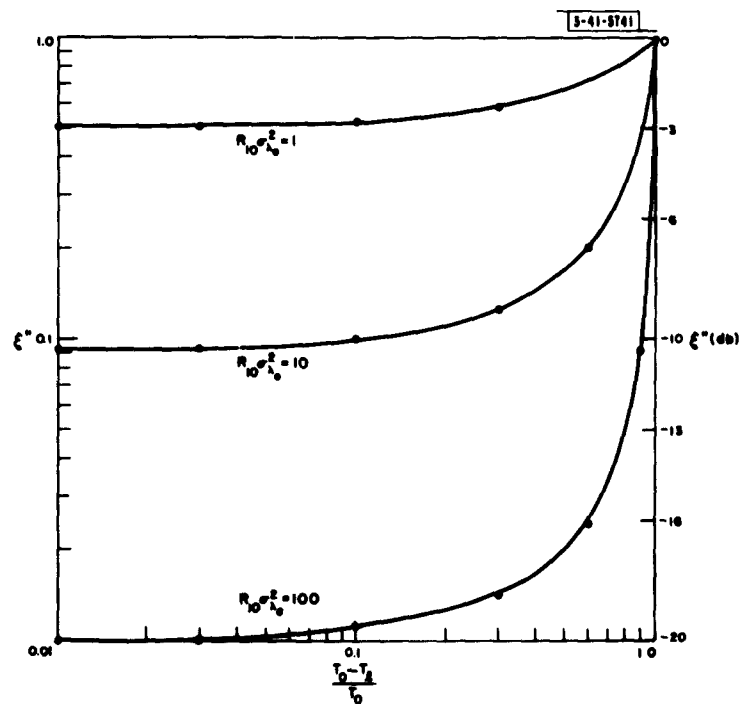Fig. 12. P_D vs P_F for optimum resolver for phase and amplitude of both returns unknown.

Fig. 13. Plot of overlap factor ξ vs overlap in received signals for resolver which neglects error. The mean interference-to-noise ratio $R_{10}\sigma^2_{\lambda_0}$ is a parameter.

we have

$$F_1'[\psi_2] = \int_0^T \upsilon(t) \, [\mu_2^*(t) - \bar{\lambda}_{12}\mu_1^*(t)] \, dt$$

$$+ \int_0^T \delta_1(t) \, \mu_1(t) \, \mu_2^*(t) \, dt$$

$$- \bar{\lambda}_{12} \int_0^T \delta_1(t) \, |\mu_1(t)|^2 \, dt + \beta_2(1 - |\bar{\lambda}_{12}|^2) \quad , \tag{61}$$

where it is noted that the terms involving β_1 add to zero. The statistics of F' that determine the envelope detector output are the mean and variance for a fixed β_2:

$$\overline{F_1'} = \beta_2(1 - |\bar{\lambda}_{12}|^2) \quad ,$$

$$\sigma_F^2 = N_0 \left[1 + R_{10}\sigma_{\lambda_0}^2 \frac{T_\ell}{T_0} - |\bar{\lambda}_{12}|^2 \left(1 + R_{10}\sigma_{\lambda_0}^2\right) \right] \quad .$$

Thus, the curves of Fig. 12 describe the P_D vs P_F characteristics of this "nonoptimum" resolver, if we replace the ξ' of Eq. (59) by ξ'', given by

$$\xi'' = \frac{[1 - |\bar{\lambda}_{12}|^2]^2}{1 + R_{10}\sigma_{\lambda_0}^2 \dfrac{T_\ell}{T_0} - |\bar{\lambda}_{12}|^2 \left[1 + R_{10}\sigma_{\lambda_0}^2\right]} \tag{62}$$

or, for large overlap, $T_\ell \approx T_0$, and

$$\xi'' \approx \frac{1 - |\bar{\lambda}_{12}|^2}{1 + R_{10}\sigma_{\lambda_0}^2} \quad . \tag{63}$$

The overlap parameter for this resolver is plotted in Fig. 13 for the case of $|\lambda_{12}| \ll 1$. Comparison with Fig. 9 for the overlap parameter for the optimum resolver shows that for large interference, the optimum is considerably superior.

We can quantitatively compare performance by asking what value of R_2 is required to achieve a value of $R_2\xi'' = R_0$, and by comparing this result to the result of the similar example in Sec. III-D, Eq. (50). Let us set

$$R_{10} = rR_2$$

and assume $|\bar{\lambda}_{12}| \ll 1$, so that by Eq. (62)

$$R_2\xi'' = R_0 \approx R_2 \frac{1}{1 + R_2 r\sigma_{\lambda_0}^2 \dfrac{T_\ell}{T_0}} \quad .$$

Note that if $R_{10}\sigma_{\lambda_0}^2 \, T_\ell/T_0 \gg 1$, we have

$$R_o \approx \frac{1}{r\sigma_{\lambda_o}^2 \frac{T_\ell}{T_o}}$$

indicating that, no matter how large R_2 may be, the second target will be resolved only when the overlap is small enough to satisfy .

$$\frac{T_\ell}{T_o} \leqslant \frac{1}{r\sigma_{\lambda_o}^2} \quad .$$

It is thus apparent that with this form of resolver, any target overlapped appreciably by a target for which the product $R_{10}\sigma_{\lambda_o}^2 > 1$ will be virtually undetectable. Thus, we can say that the uncertainty in the ambiguity function side-lobe structure sets a rather definitive limit on the difference in radar cross section of two overlapping targets that can be resolved by any detector. Furthermore, if the possibility of signal distortion is ignored in constructing the resolver, the limit on resolvability is quite insensitive to the radar sensitivity (signal-to-noise ratio achieved).

B. The Resolution Capability of the Usual "Matched" Filter

We conclude this investigation by inquiring about the degree of resolution obtainable if one does "nothing special"; the radar processor consists only of a filter "matched" to $\mu_2(t)$, which for a Gaussian cross section is

$$F_2[\psi_2(t)] = |F_2'[\psi_2(t)]| \quad , \tag{64}$$

where

$$F_2'[\psi_2] = \int_0^T \zeta(t) \, \mu_2^*(t) \, dt \quad . \tag{65}$$

The output for $\zeta(t) = \upsilon(t) + [\beta_1 + \delta_1(t)] \, \mu_1(t) + \beta_2 \mu_2(t)$ is

$$F_2'[\psi_2] = \int_0^T \upsilon(t) \, \mu_2^*(t) \, dt + \beta_1 \bar{\lambda}_{12} + \int_0^T \delta_1(t) \, \mu_1(t) \, \mu_2^*(t) \, dt + \beta_2 \quad . \tag{66}$$

The question that we framed was what happens if we do "nothing special" about resolution. Obviously, one of the first things that may happen is that the response of the filter matched to $\mu_2(t)$ and to Signal 1 will raise the false-alarm rate unless the threshold is appropriately adjusted. To see this, note that from Eq. (66), the variance of F_2' is

$$\sigma_{F_2}^2 = N_o \left[1 + R_{10} \left(\frac{|\bar{\lambda}_{12}|^2}{2} + \sigma_{\lambda_o}^2 \frac{T_\ell}{T_o} \right) \right] \quad ,$$

indicating that the threshold, for a constant false-alarm rate, must be raised by (in terms of voltage) the ratio

$$\tau' = \tau \left[1 + R_{10} \left(\frac{|\bar{\lambda}_{12}|^2}{2} + \sigma_{\lambda_o}^2 \frac{T_\ell}{T_o} \right) \right] \quad , \tag{67}$$

(where τ is the proper threshold setting for nonoverlapping targets) to prevent the interference from target one altering P_F.

Since, for β_2 fixed,

$$\overline{F'_2} = \beta_2 \quad ,$$

we see that

$$\frac{\overline{(F'_2)^2}}{\sigma_L^2} = \frac{R_2}{1 + R_{10}\left(\dfrac{|\overline{\lambda}_{12}|^2}{2} + \sigma_{\lambda_0}^2 \dfrac{T_\ell}{T_0}\right)}$$

from which the potential performance of the simple matched filter can be determined from the curves of Fig. 12, replacing ξ' by ξ''', given by

$$\xi''' = \frac{1}{1 + R_{10}\dfrac{|\overline{\lambda}_{12}|^2}{2} + R_{10}\sigma_{\lambda_0}^2 \dfrac{T_\ell}{T_0}} \quad . \tag{68}$$

It is apparent that if $|\overline{\lambda}_{12}|^2 \ll 1$ in Eq. (62), and also $|\overline{\lambda}_{12}|^2 \ll \sigma_{\lambda_0}^2 T_\ell/T_0$ in Eq. (68) (the error in λ predominates), the simple filter resolves as well as the filter which ignores only the error $\delta_1(t)$ examined in Sec. IV-D. However, it is apparent from Eq. (67) that the requirement for the filter mentioned in Sec. IV-D to be little affected by $|\overline{\lambda}_{12}|$, is $|\overline{\lambda}_{12}|^2 \ll 1$ if the targets overlap appreciably; for the matched filter it is necessary that $|\overline{\lambda}_{12}|^2 \ll 2\sigma_{\lambda_0}^2 T_\ell/T_0$, which is much less than unity. For example, for $T_\ell \approx T_0$, we can write

$$\frac{\xi''(\text{nonoptimum})}{\xi'''(\text{matched})} \approx (1 - |\overline{\lambda}_{12}|^2)\left[1 + \frac{R_{10}|\overline{\lambda}_{12}|^2}{2\left(1 + R_{10}\sigma_{\lambda_0}^2\right)}\right] \tag{69}$$

or, for $R_{10}\sigma_{\lambda_0}^2 \gg 1$,

$$\frac{\xi''(\text{nonoptimum})}{\xi'''(\text{matched})} \approx (1 - |\overline{\lambda}_{12}|^2)\left(1 + \frac{|\overline{\lambda}_{12}|^2}{2\sigma_{\lambda_0}^2}\right)$$

which for $\sigma_{\lambda_0}^2 \ll 1/2$ and $|\overline{\lambda}_{12}|^2 \ll 1$, as we would expect, is clearly greater than unity. In regions where the "no-error" side lobes of the ambiguity function are large compared to the effect of errors, the filter of the previous section is markedly superior. This result is not unexpected, since the major difference is that the previous filter optimally provides for the "no-error" ambiguity function, but ignores the errors, while the simple matched filter ignores both.

V. SUMMARY AND CONCLUSIONS

A. Summary of Results

W⌐ have explored limitations upon the ability of a radar to detect a target in the presence of an overlapping target. We have investigated the effect of the inevitable small signal distortions, as well as the effect of lack of a priori knowledge about the parameters of the two targets.

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We have assumed that the signals returned from the target are randomly distorted by a stationary complex Gaussian process $\delta_1(t)$ so that the distorted signal is of the form

$$\psi_1(t) = [\beta_1 + \delta_1(t)]\,\mu_1(t) \quad ,$$

where $\beta_1\mu_1(t)$ is the undistorted signal. The $\delta(t)$ process is assumed to have mean zero and variance

$$|\delta_1(t)|^2 = 4E_1\sigma_\delta^2 \quad ,$$

where $E_1 = |\beta_1|^2/2$ is the energy in the real part of the return in the absence of error (or the expected energy if the cross section is random).

The effect of this distortion is to contribute an uncertainty to the shape of the radar ambiguity function.

We showed that for wide-band distortion of coded pulses (see Sec. II-A for examples), assuming a flat spectral density for the distortion, the variance in the ambiguity function in the region near the main lobe $\sigma_{\lambda_0}^2$ was related to the signal distortion variance by

$$\sigma_{\lambda_0}^2 = \frac{\sigma_\delta^2}{T_0 W_D} \quad , \tag{23a}$$

where T_0 is the pulse duration and W_D, as previously pointed out, is the distortion bandwidth. The variance in the ambiguity function varies with the ratio of the time elapsed from main lobe T_ℓ to the duration of the pulse T_0

$$\sigma_\lambda^2 = \sigma_{\lambda_0}^2 \frac{T_\ell}{T_0} \quad , \tag{23b}$$

where we can interpret T_ℓ/T_0 for the resolution problem as the fraction of the pulse duration during which the signals overlap.

We found that for pulse-train signals, and for errors correlated within individual pulses but uncorrelated pulse to pulse

$$\sigma_\lambda^2 = \frac{\sigma_\delta^2}{N}\,\bar{\lambda}_{12} \quad , \tag{25}$$

where $\bar{\lambda}_{12}$ is the value of the "no-error" ambiguity function.

We derived the optimum resolver and calculated its performance. We found that a convenient method for specifying the performance of any of the resolvers investigated was a factor ξ, which we designated an "overlap factor," since it defines the fractional increase in signal-to-noise ratio $R_2 = 2E_2/N_0$, required to compensate for the presence of the interfering target.

We found that, for the optimum resolver as specified by Eq. (40) when both signals are exactly known (only the presence of the second is in doubt),

$$\xi = \frac{1 + R_1\sigma_{\lambda_0}^2\left(\dfrac{T_0 - T_\ell}{T_0}\right)}{1 + R_1\sigma_{\lambda_0}^2} \tag{48}$$

for coded pulse (see Fig. 9) and

$$\xi = \frac{1 + R_1 \sigma_\lambda^2 \left(\frac{1 - \bar{\lambda}_{12}}{\bar{\lambda}_{12}}\right)}{1 + R_1 \left(\frac{\sigma_\lambda^2}{\bar{\lambda}_{12}}\right)} \qquad (49)$$

for pulse trains. We found that the value of R_2 required for $P_D = 0.9$, $P_F = 10^{-9}$ was approximately

$$R_2 = \frac{50}{\frac{T_o - T_\ell}{T_o}} \qquad (50)$$

when the product of the ratio of the target cross sections and $\sigma_{\lambda_o}^2$ is of the order of unity or greater. Also we found that for the target cross sections unknown and assumed to be complex random variables, so that the mean signal-to-noise ratio of Target 1 R_{10}, the optimum resolver is an envelope detector, operating on a circuit specified by Eq. (54), for which

$$\xi' = \frac{1 - |\bar{\lambda}_{12}| + R_{10}\sigma_{\lambda_o}^2 \left(\frac{T_o - T_\ell}{T_o}\right)}{1 + R_{10}\sigma_{\lambda_o}^2} \qquad (59)$$

for coded pulses, with the modification for pulse trains suggested by comparison of Eqs. (48) and (49). The curves of Fig. 12 are applicable for R_2'.

We also investigated the performance of two nonoptimum resolvers. The first is the filter that Eq. (54) indicates would be optimum in the absence of error (we assumed in the filter design that $\sigma_\lambda^2 \equiv 0$). We then derived the performance in the presence of errors, and found that

$$\xi'' = \frac{[1 - |\bar{\lambda}_{12}|^2]^2}{1 + R_{10}\sigma_{\lambda_o}^2 \frac{T_\ell}{T_o} - |\bar{\lambda}_{12}|^2 \left[1 + R_{10}\sigma_{\lambda_o}^2\right]} \quad . \qquad (62)$$

We indicated from this result that for a large overlapping target resolution can be achieved only by reduced overlap, and not by increased radar sensitivity (to a first order).

Secondly, we analyzed the performance of the usual "matched" filter (one that is optimum for nonoverlapping signals). We pointed out that the presence of a first signal will cause false alarms if the threshold is not properly varied, according to Eq. (67), relative to the ambiguity function of the first target. If the threshold is properly varied, then

$$\xi''' = \frac{1}{1 + \frac{R_{10}|\bar{\lambda}_{12}|^2}{2} + R_{10}\sigma_{\lambda_o}^2 \frac{T_\ell}{T_o}} \quad . \qquad (68)$$

B. Conclusions

The results of the investigation, notably the forms of Eqs. (48), (49) and (59), indicate that the fundamental limitations on the resolvability of targets are: (1) the inevitable unknowable departure of the shape of the received signal from that expected, and (2) our lack of a priori knowledge about the targets.

The first limitation appears to be the more restrictive since the ambiguity function uncertainty appears in the performance equations always in the form of $R_{10}\sigma_{\lambda_o}^2$, which for large interfering cross sections is likely to be considerably greater than unity.[†] For the distortion model assumed, the uncertainty in signal parameters, at least as indicated in the difference between Eqs. (48) and (59) above, manifests itself only in the factor $1 - |\bar{\lambda}_{12}|^2$, so that outside the main lobe of most ambiguity functions, the effect is certainly negligible,[‡] regardless of the cross section of the interfering target.

The hardware cost of realizing a bank of optimum resolvers, which would be required when signal parameters are unknown, would probably be exorbitant. Looking toward simpler, non-optimum resolvers, an interesting comparison can be obtained by comparing the performance of the three resolvers [Eqs. (59), (62) and (68)] for random target cross sections, for the situation in which interference predominates thermal noise $[R_{10}\sigma_{\lambda_o}^2 \gg 1]$, and the "no-error" ambiguity function magnitude is negligible compared to unity. The quantity which seems to shed most light upon the performance is $R_2\xi$, which, letting the target cross-section ratio be represented by $r = R_{10}/R_2$ is then approximately

$$R_2\xi' \approx \frac{1}{r\sigma_{\lambda_o}^2}\left[1 + R_2 r\sigma_{\lambda_o}^2\left(\frac{T_o - T_\ell}{T_o}\right)\right]$$

for the optimum resolver;

$$R_2\xi'' = \frac{1}{r\sigma_{\lambda_o}^2\dfrac{T_\ell}{T_o}}$$

for the optimum resolver for a distortion-free signal and

$$R_2\xi''' = \frac{1}{r\sigma_{\lambda_o}^2}\left(\frac{1}{\dfrac{|\bar{\lambda}_{12}|^2}{2\sigma_{\lambda_o}^2} + \dfrac{T_\ell}{T_o}}\right)$$

for the simple matched filter, properly thresholded. Note that for large target overlaps, the quantity $R_2\xi$ is independent of the radar sensitivity (to a first order, at least) and the targets are truly unresolvable. However, the optimum detector resolution capability increases rapidly with decreasing overlap, while the other resolvers improve only slowly until the overlap is nearly zero.

The superiority of performance of the optimum resolver is thus clear. The choice between the two nonoptimum filters revolves about the relative amplitudes of the known and unknown components of the ambiguity function. It would appear that with reasonable sophistication in design,

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[†] Indeed, if it isn't, thermal noise dominates the detection problem, and the resolution problem is virtually nonexistent.

[‡] Helstrom's[9] results, which include the effects of unknown signal arrival time, would seem to indicate an increasingly stronger dependence on $|\bar{\lambda}_{12}|$ as the number of unknown signal parameters increases. However, his results also indicate that when $|\bar{\lambda}_{12}|^2 \ll 1$, the effect of unknown parameters is negligible in an optimum resolver.

it should be possible to render $|\bar{\lambda}_{12}|$ small enough over most of the ambiguity function space so that errors will predominate. If this is the case, the simple "matched filter," properly thresholded, would appear to be the preferred "nonoptimum" detector for reasons of simplicity.

ACKNOWLEDGMENTS

REFERENCES

1. C.E. Cook, "Pulse Compression-Key to More Efficient Radar Transmission," Proc. IRE 48, 310 (March 1960).

2. J.R. Klauder, et al., "The Theory and Design of Chirp Radars," Bell System Tech. J. 39, 745 (July 1960).

3. E.L. Key, E.N. Fowle and R.D. Haggarty, "A Method of Pulse Compression Employing Nonlinear Frequency Modulation," Technical Report No. 207 [U], Lincoln Laboratory, M.I.T. (13 August 1959), ASTIA 312903.

4. J.S. Hall, Ed., Radar Aids to Navigation, Radiation Laboratory Series, M.I.T. (McGraw-Hill, N.Y., 1947) Vol.2, pp. 150-151.

5. P.M. Woodward, Probability and Information Theory (Pergamon Press, New York, 1953).

6. E.J. Kelly, I.S. Reed, "Some Properties of Stationary Gaussian Processes," Technical Report No. 157 [U], Lincoln Laboratory, M.I.T. (5 June 1957), ASTIA 150864.

7. J.L. Doob, Stochastic Processes (John Wiley, New York, 1953).

8. C.W. Helstrom, "The Resolution of Signals in White, Gaussian Noise," Proc. IRE 43, 1111 (September 1955).

9. _____, "The Resolution of Pulsed Signals," Research Report 8-1259-R6 Westinghouse Research Laboratories (18 July 1957).

10. _____, Statistical Theory of Signal Detection (Pergamon Press, New York, 1960).

11. W.Mc. Siebert, "A Radar Detection Philosophy," Trans. IRE, PGIT IT-2, 204 (September 1956).

12. _____, "Studies of Woodward's Uncertainty Function," Quarterly Progress Report, Research Laboratory of Electronics, M.I.T. (April 1958).

13. E.J. Kelly, "The Radar Measurement of Range, Velocity and Acceleration," 47G-0013 [U], Lincoln Laboratory, M.I.T. (6 January 1961) ASTIA 249468, H-243.

14. B.R. Mayo, P.W. Howells and W.B. Adams, "Generalized Linear Radar Analysis," Microwave J. 4, 79 (August 1961).

15. E.N. Fowle, E.J. Kelly and J.A. Sheehan, "A Radar System Performance in a Dense-Target Environment," 1961 IRE International Convention Record, Part 4, pp. 136-145.

16. N.J. Nilsson, "On the Optimum Range Resolution of Radar Signals in Noise," Trans. IRE, PGIT IT-7, 245 (October 1961).

17. G.W. Preston, "The Theory of Pulsed Signal Measurements," Final Report 779-207-12, General Atronics Corporation (20 August 1960).

18. R.H. Barker, "Group Synchronizing of Binary Digital Systems," Communication Theory, Willis Jackson, Ed. (Academic Press, 1953), pp. 273-287.

19. J.B. Resnick, "High Resolution Waveforms Suitable for a Multiple Target Environment," M.S. Thesis, Department of Electrical Engineering, M.I.T. (May 1962).

20. C. Polk, "Transient Behavior of Aperture Antennas, Proc. IRE 48, 1281 (July 1960).

21. J.L. Allen, et al., "Phased Array Radar Studies, 1 July 1960 to 1 July 1961, Technical Report No. 236 [U], Lincoln Laboratory, M.I.T. (13 November 1961), ASTIA 271723.

22. W.B. Davenport, W.L. Root, An Introduction to the Theory of Random Signals (McGraw-Hill, New York, 1958).

23. H. Cramer, Mathematical Methods of Statistics (Princeton University Press, Princeton 1946).

24. W.W. Peterson, T.G. Birdsall and W.C. Fox, "The Theory of Signal Detectability," Trans. IRE, PGIT IT-7, 171 (September 1954).

25. S.O. Rice, "Mathematical Analysis of Random Noise," Noise and Stochastic Process, Nelson Wax, Ed. (Dover Publications, New York, 1954).

26. J. Ruze, "Physical Limitations on Antennas," Technical Report No. 248, Research Laboratory of Electronics, M.I.T. (20 August 1956, ASTIA 265867.

27. R. Manasse, "The Application of the Theory of Signal Detectability to Signals with Unknown Polarization and Phase," Group Report 32-25, Lincoln Laboratory, M.I.T. (20 August 1956), ASTIA 236167.